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Mecánica de Medios Continuos y Teoría de Estructuras

Resistencia de Materiales

EL SÓLIDO ELÁSTICO

(Ley de comportamiento)

CONTENIDO DE LA ASIGNATURA

BLOQUE TEMATICO: ELASTICIDAD Y RESISTENCIA DE MATERILES

CAPÍTULO 1: INTRODUCCIÓN A LA ELASTICIDAD Y LA RESISTENCIA DE MATERIALES

CAPÍTULO 2: EL SÓLIDO ELÁSTICO.

CAPÍTULO 3: CRITERIOS DE PLASTIFICACIÓN Y DE ROTURA

CAPÍTULO 4: RESISTENCIA DE MATERIALES. CONCEPTOS BÁSICOS

CAPÍTULO 5: TRACCIÓN Y COMPRESIÓN

CAPÍTULO 6: FLEXIÓN PLANA ELÁSTICA.

CAPÍTULO 7: INTRODUCCIÓN AL CÁLCULO PLÁSTICO

CAPÍTULO 8: FLEXO-COMPRESIÓN DESVIADA

CAPÍTULO 9: TORSIÓN

CAPÍTULO 10: POTENCIAL ELÁSTICO DE BARRAS. MÉTODOS ENERGÉTICOS

CAPÍTULO 11: INESTABILIDAD DE BARRAS PRISMÁTICAS. PANDEO

Resistencia de Materiales

EL SÓLIDO ELÁSTICO

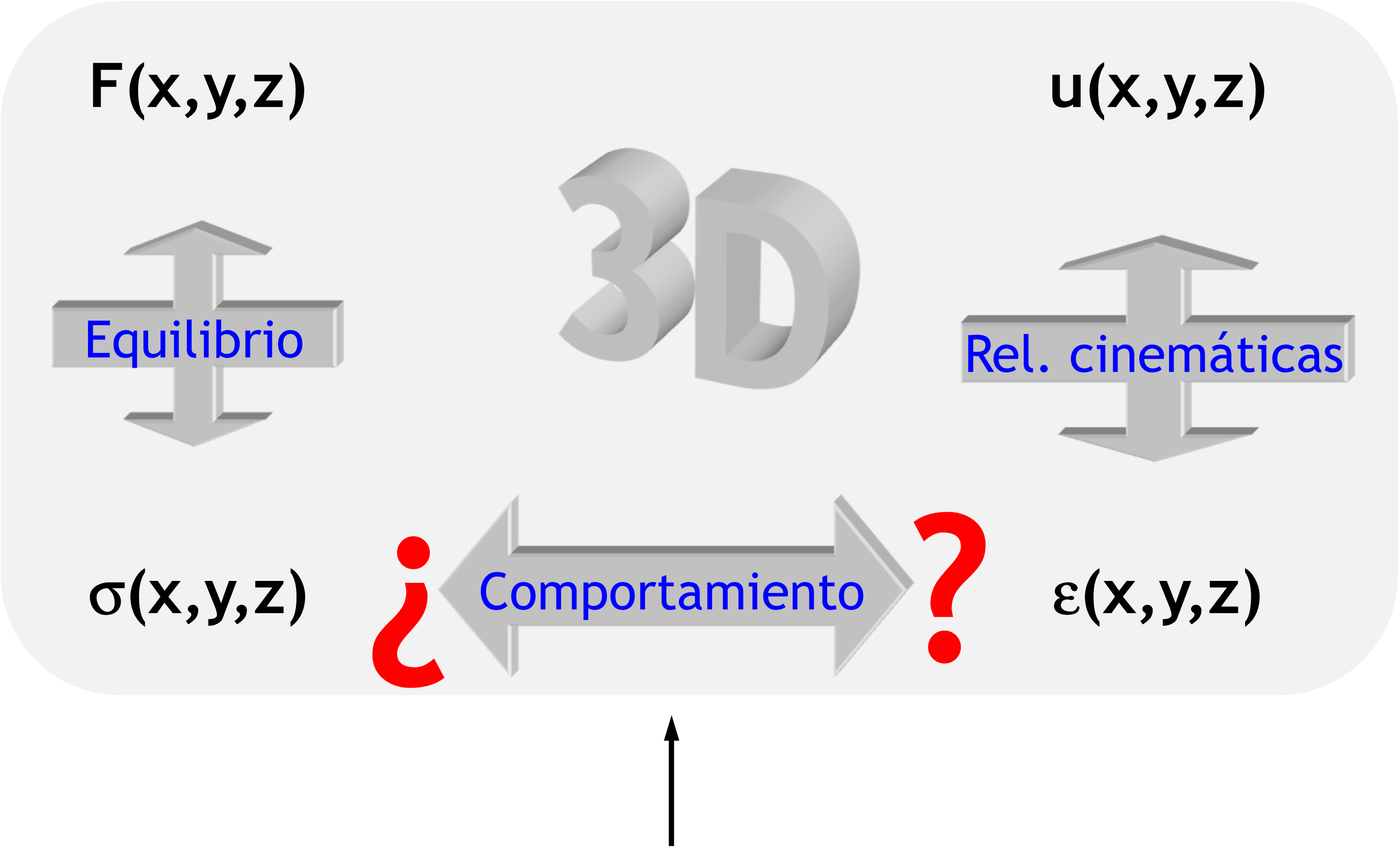
(Ley de comportamiento)

Mecánica de Medios Continuos y Teoría de Estructuras

- Introducción.
- El ensayo de tracción monoaxial.
- Ley de Hooke generalizada.
- Módulo de cizalladura.
- Ley de comportamiento en unas coordenadas cualesquiera.
- El problema elástico.
- Condiciones de contorno.
- El problema térmico.
- Energía de deformación.
- Principio de Saint-Venant.

Introducción

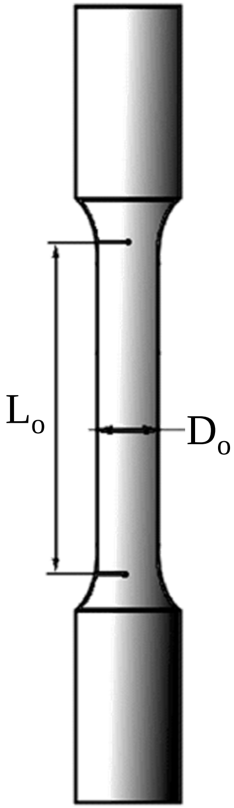
El problema elástico



EXPERIMENTACIÓN

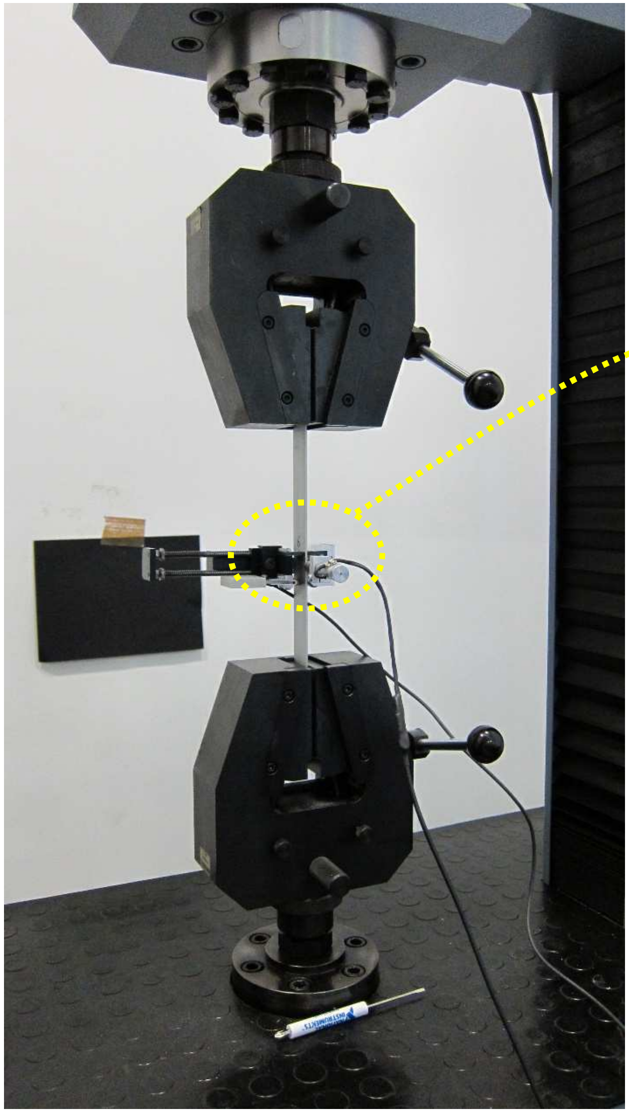
Ensayo de tracción

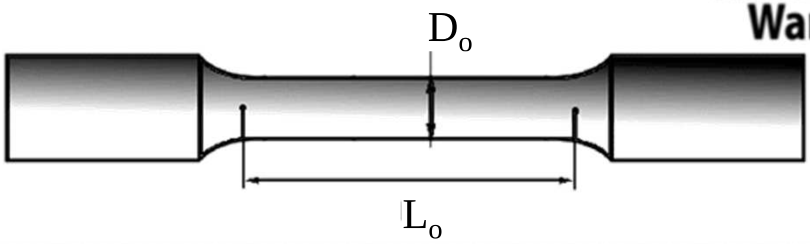
** máquina de ensayos del Laboratorio de Resistencia de Materiales - UMA*

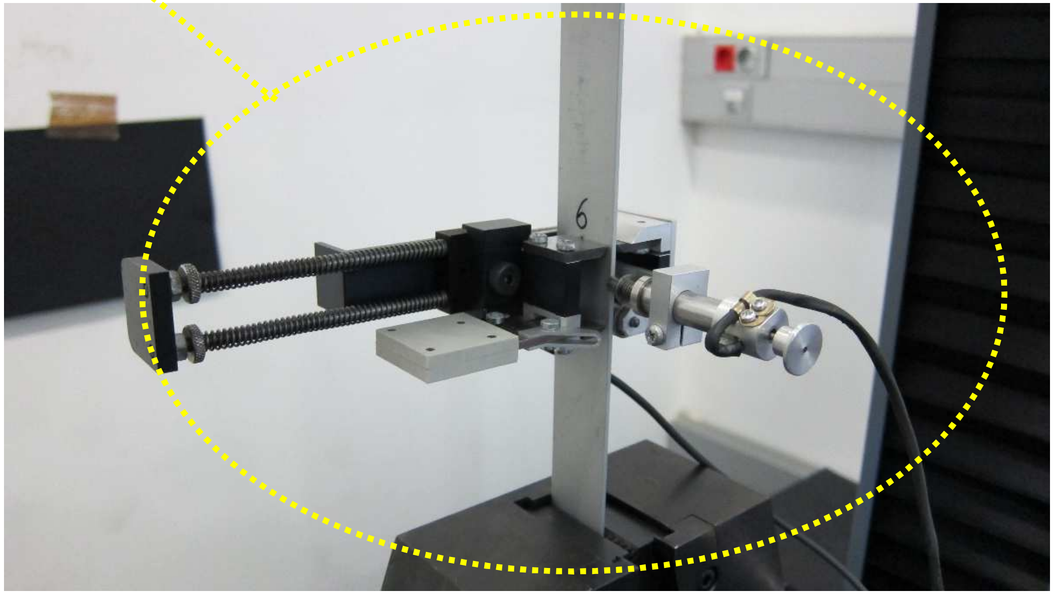


** probeta cilíndrica tipo – recreación gráfica*

Ensayo de tracción

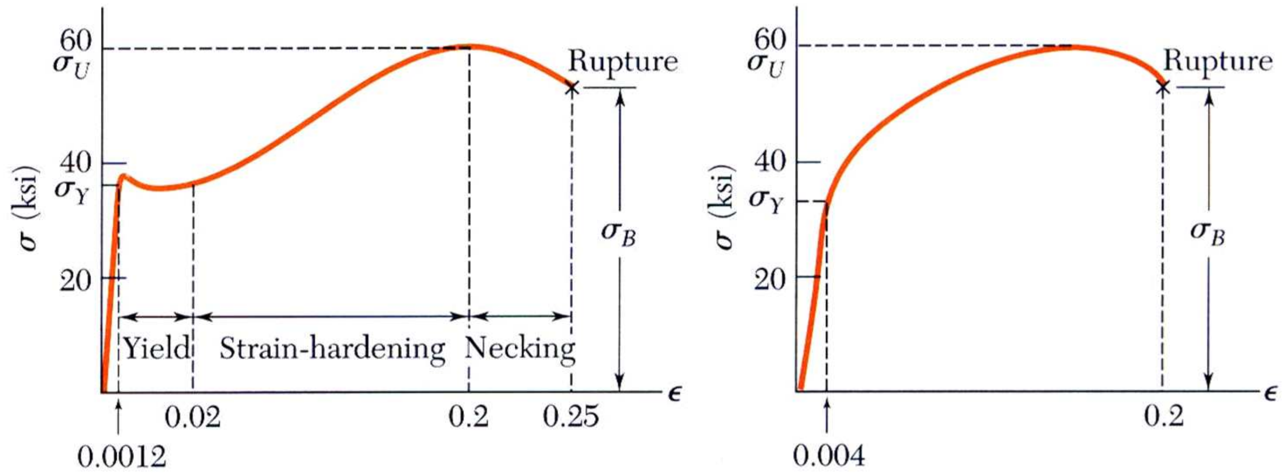




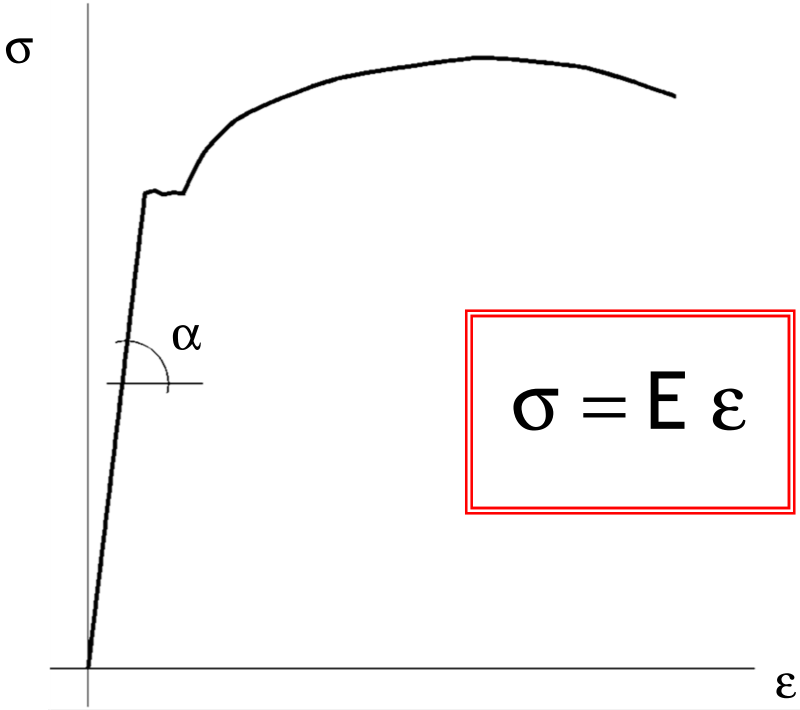


* *Vista detallada del extensómetro electrónico fijado a la probeta*

Ensayo de tracción



(a) Low-carbon steel (b) Aluminum alloy



$$\mathrm{tg}(\alpha)=E > 0$$

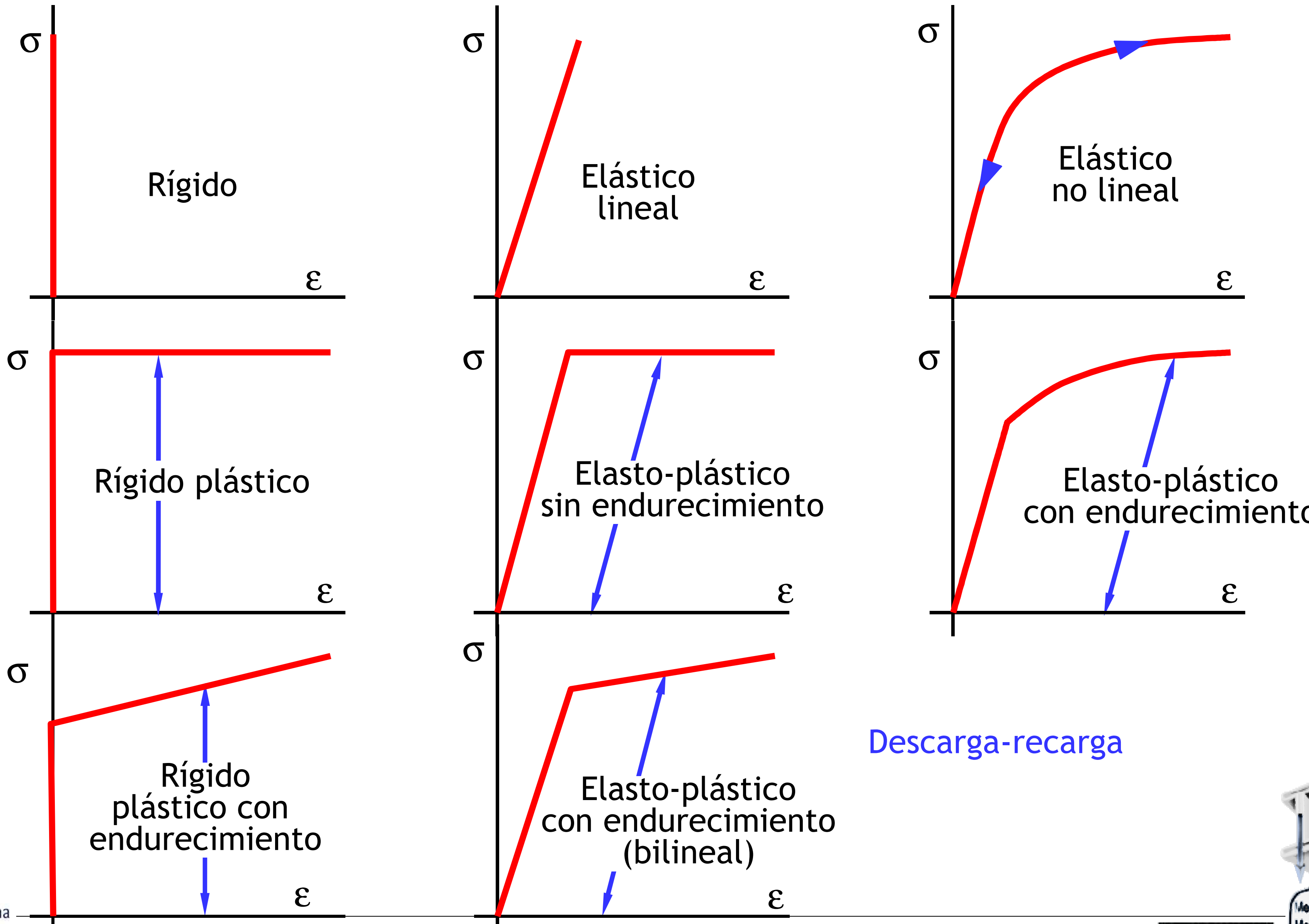
$$\sigma = E\,\varepsilon$$

Thomas Young
13/06/1773 (Somerset, Inglaterra)-
-10/05/1829 (Londres)

** fuente: wikipedia*

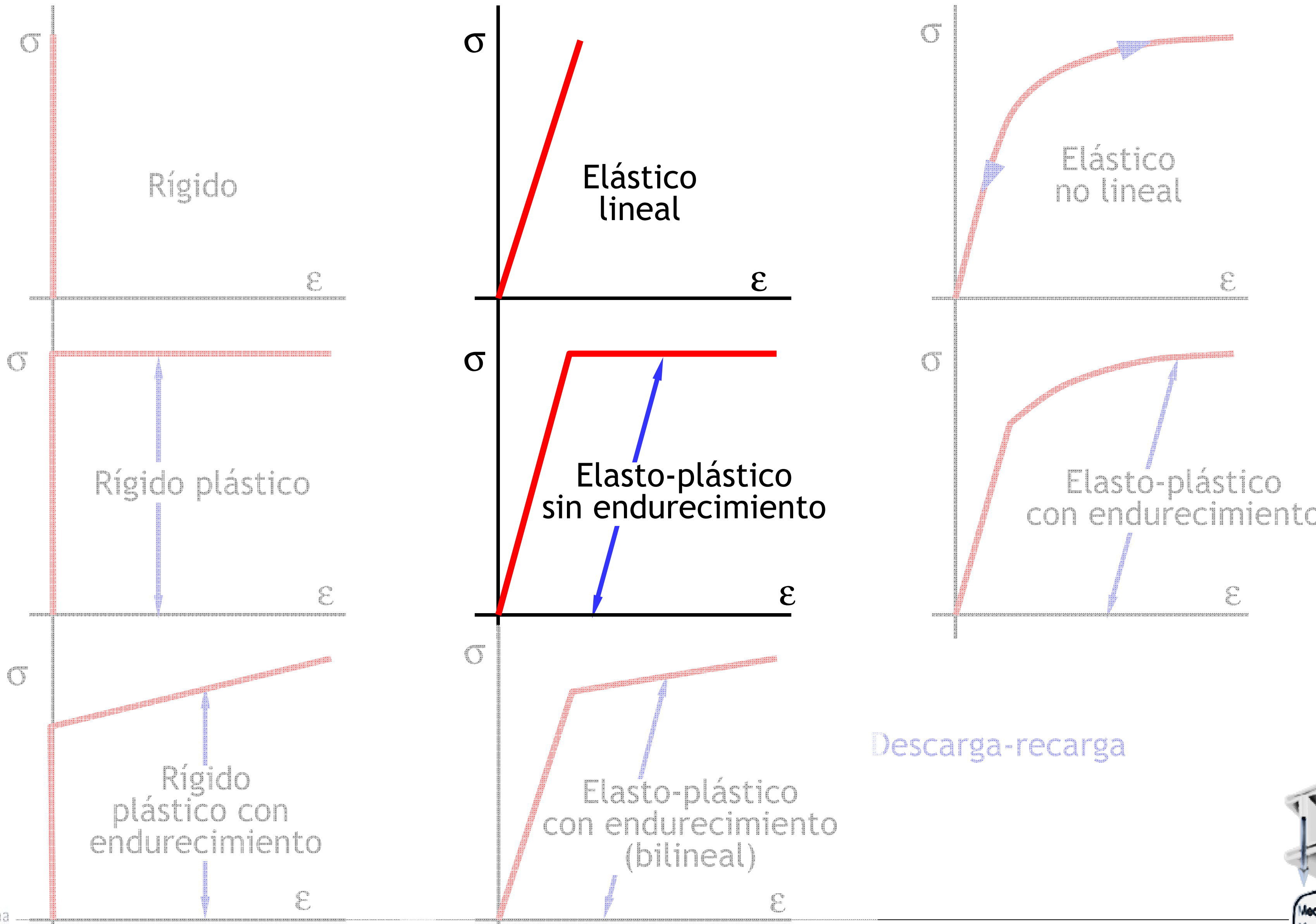
Ensayo de tracción

Diferentes **modelos** de comportamiento

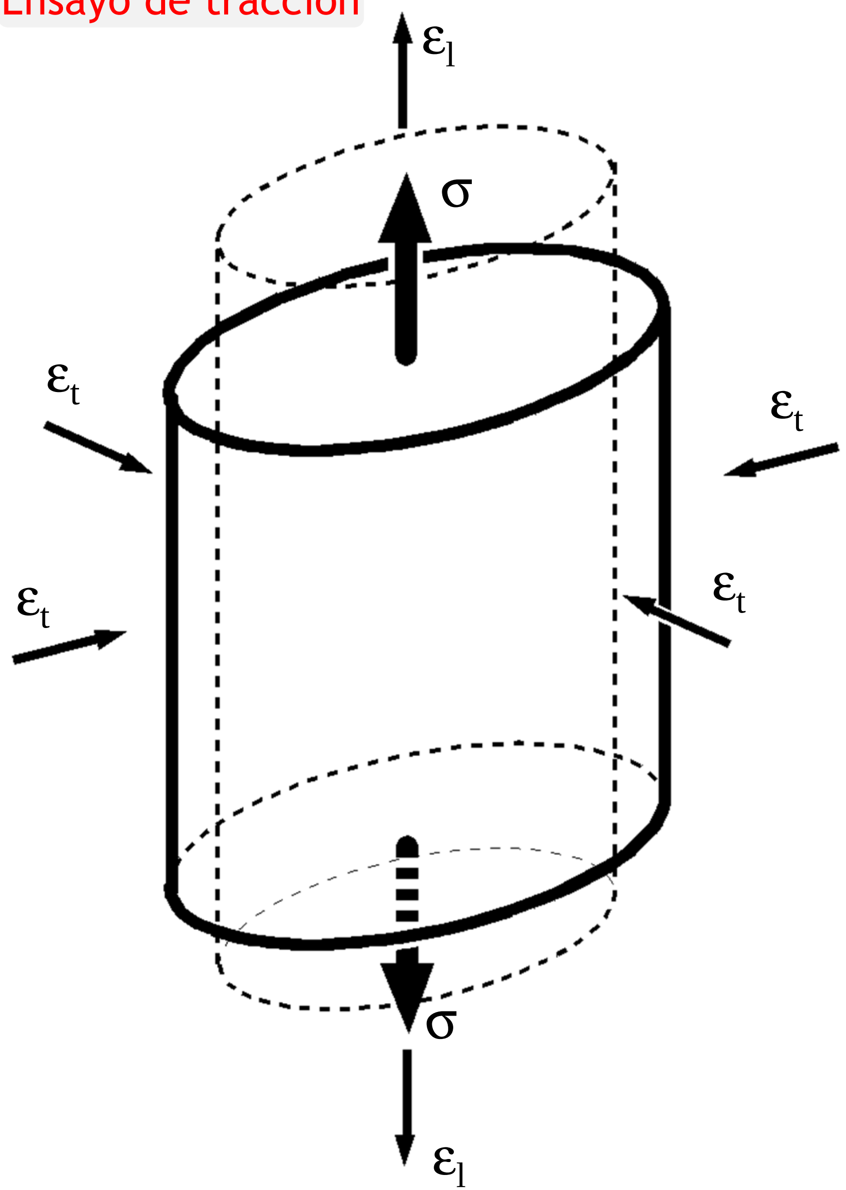


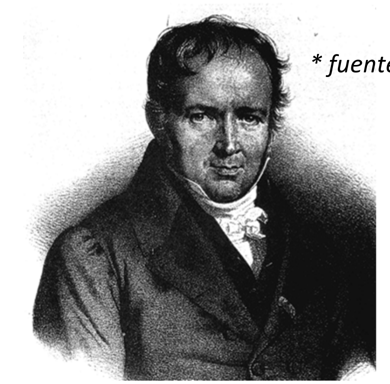
Ensayo de tracción

Diferentes **modelos** de comportamiento



Ensayo de tracción





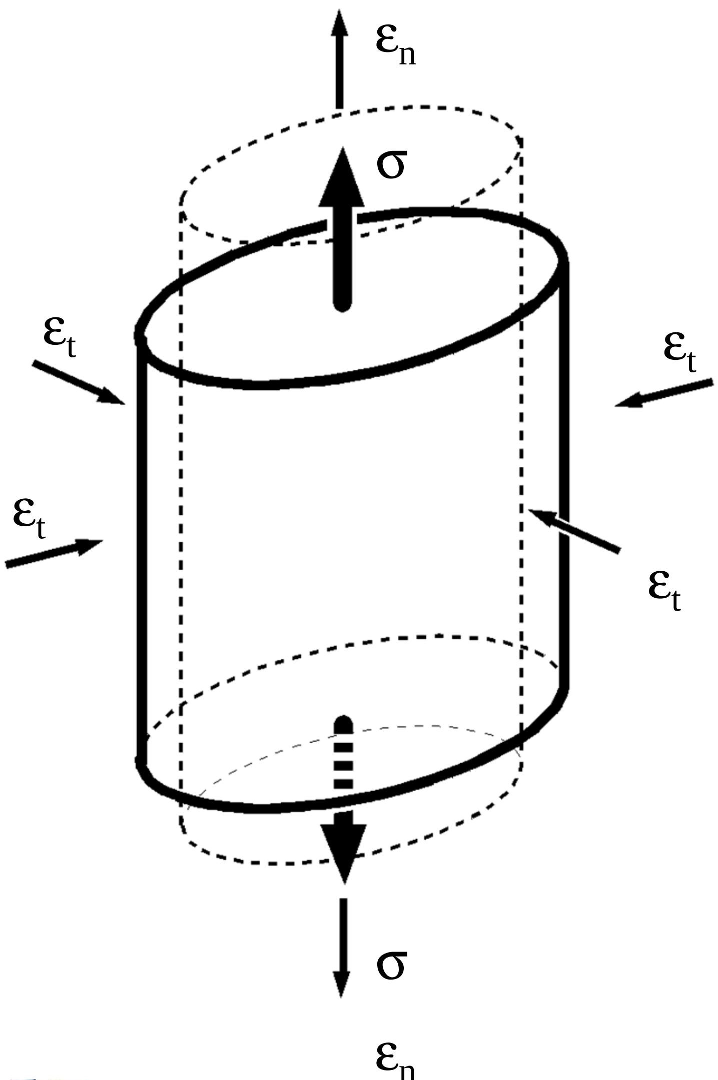
* *fuente: wikipedia*

Simeón Denis Poisson
21/06/1781 (Pithiviers, Francia)-
-25/04/1840 (Sceaux, Francia)

$$\boxed{\varepsilon_t = -\nu\, \varepsilon_l}$$

Nota: $\nu > 0$

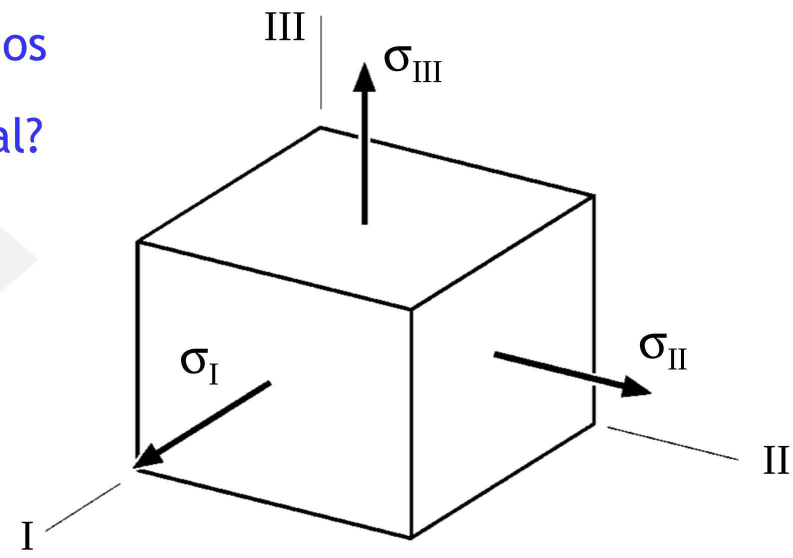
Ley de Hooke generalizada



¿Cómo pasamos al caso tridimensional?

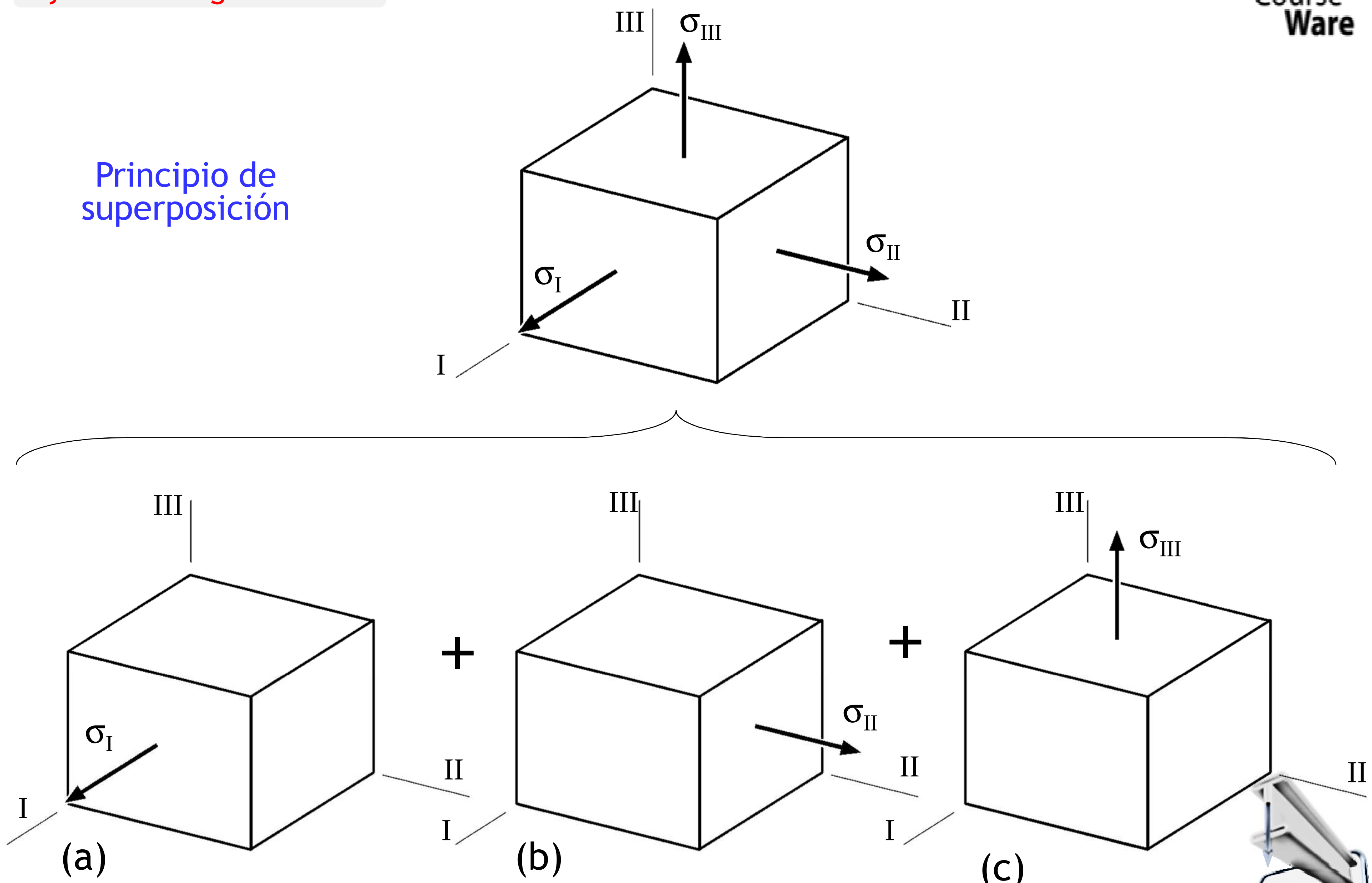
* *fuente: wikipedia*

Robert Hooke
18/07/1635 (Freshwater, Inglaterra)-
-03/03/1703 (Londres)

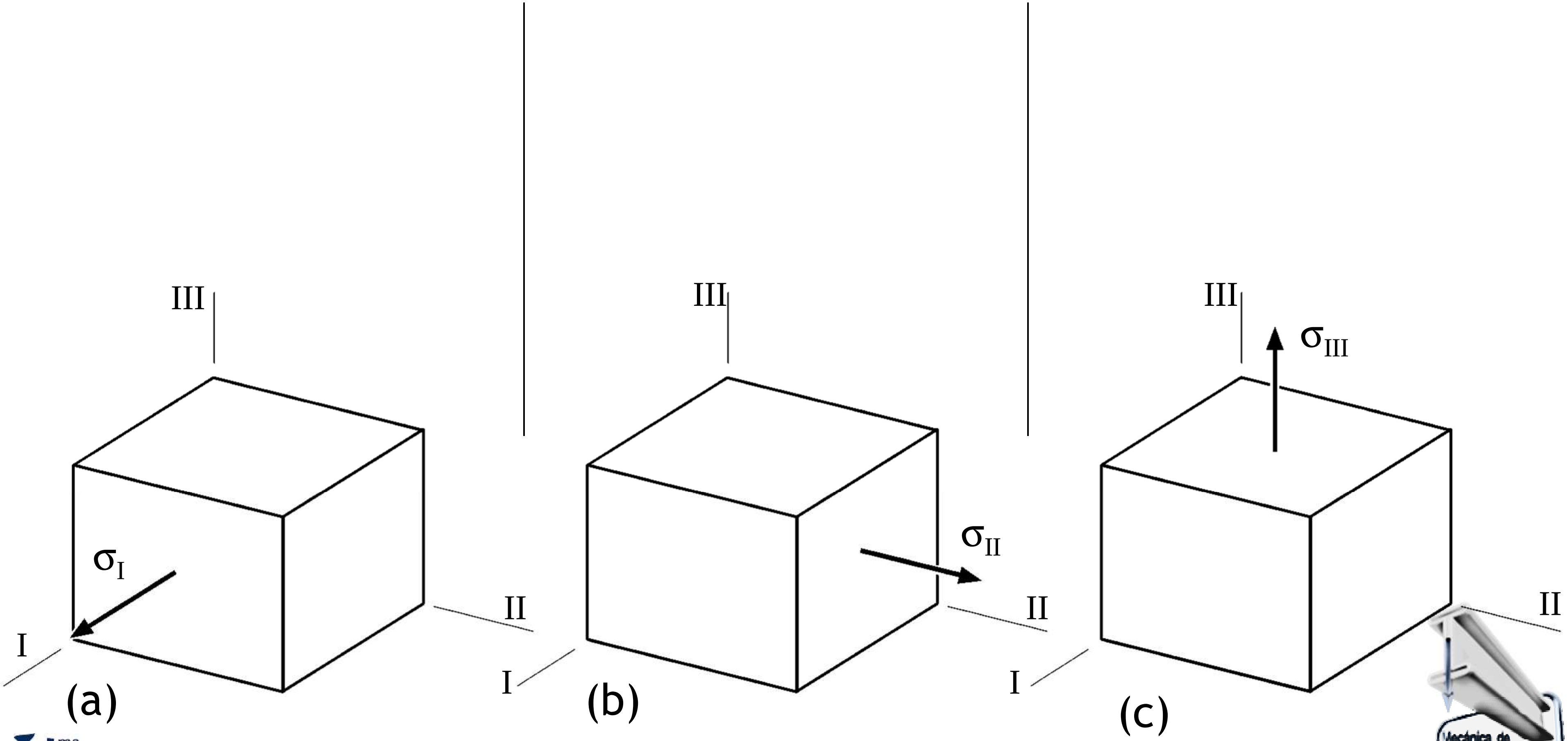


Ley de Hooke generalizada

Principio de superposición



Ley de Hooke generalizada



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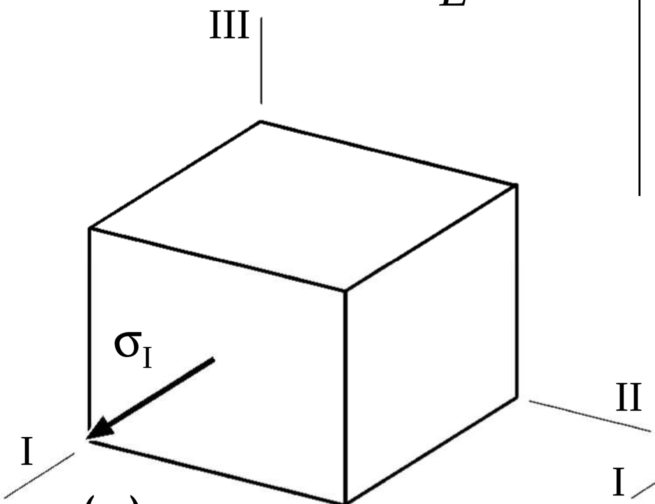
Ley de Hooke generalizada

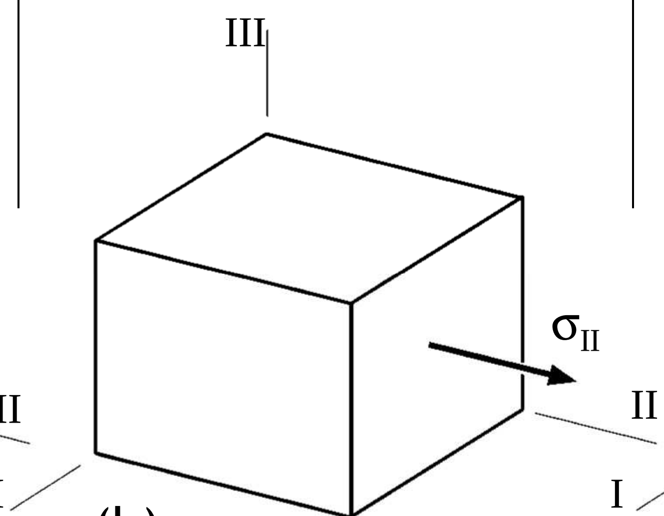
$$\varepsilon_I = \frac{\sigma_I}{E}$$

$$\varepsilon_{II} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

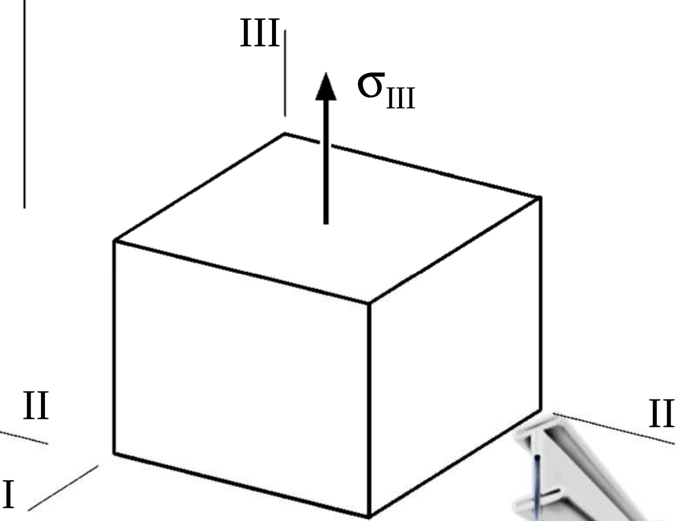
$$\varepsilon_{III} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$



(a)



(b)



(c)

Ley de Hooke generalizada

$$\varepsilon_I = \frac{\sigma_I}{E}$$

$$\varepsilon_{II} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_{III} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

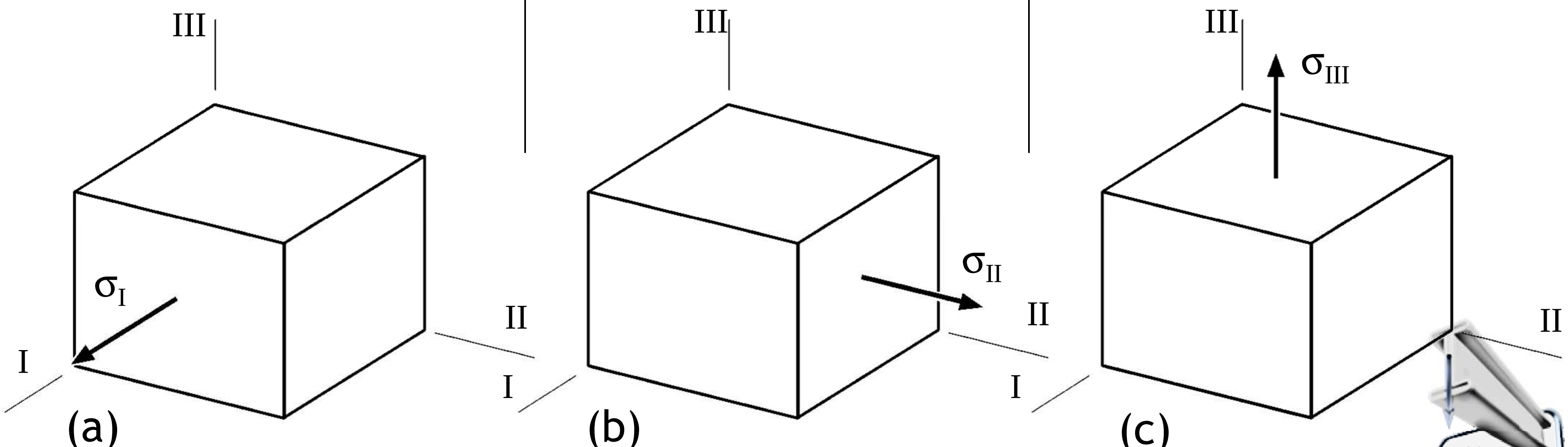
$$\varepsilon_{II} = \frac{\sigma_{II}}{E}$$

$$\varepsilon_{III} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E}$$

$$\varepsilon_{II} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E}$$

$$\varepsilon_{III} = \frac{\sigma_{III}}{E}$$



Ley de Hooke generalizada

$$\varepsilon_I = \frac{\sigma_I}{E}$$

$$\varepsilon_{II} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_{III} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

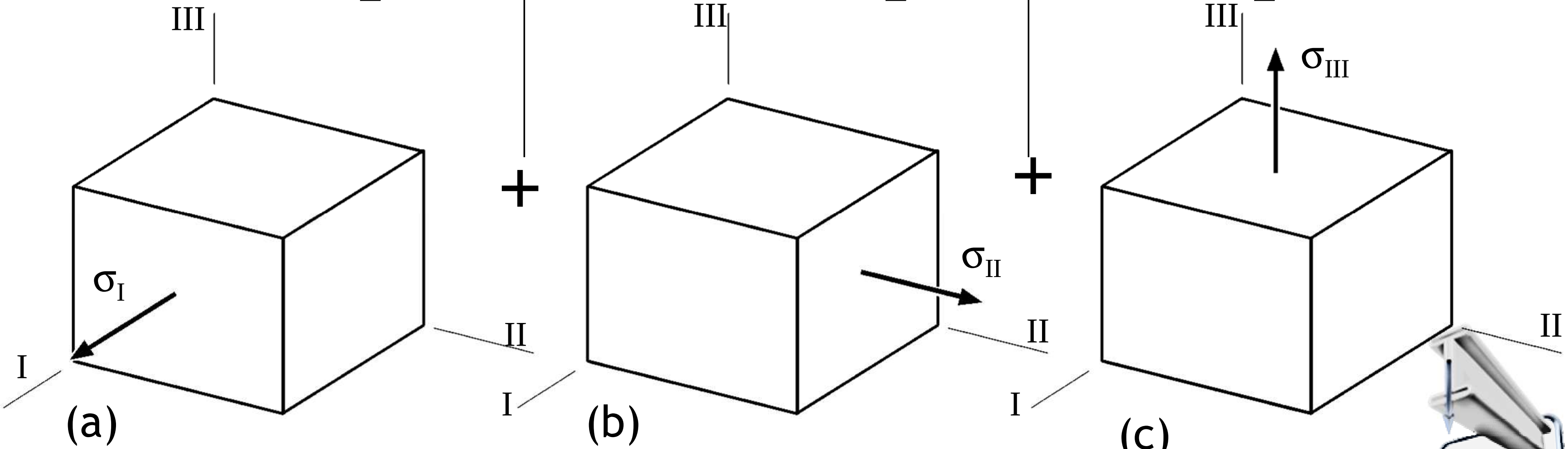
$$\varepsilon_{II} = \frac{\sigma_{II}}{E}$$

$$\varepsilon_{III} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E}$$

$$\varepsilon_{II} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E}$$

$$\varepsilon_{III} = \frac{\sigma_{III}}{E}$$



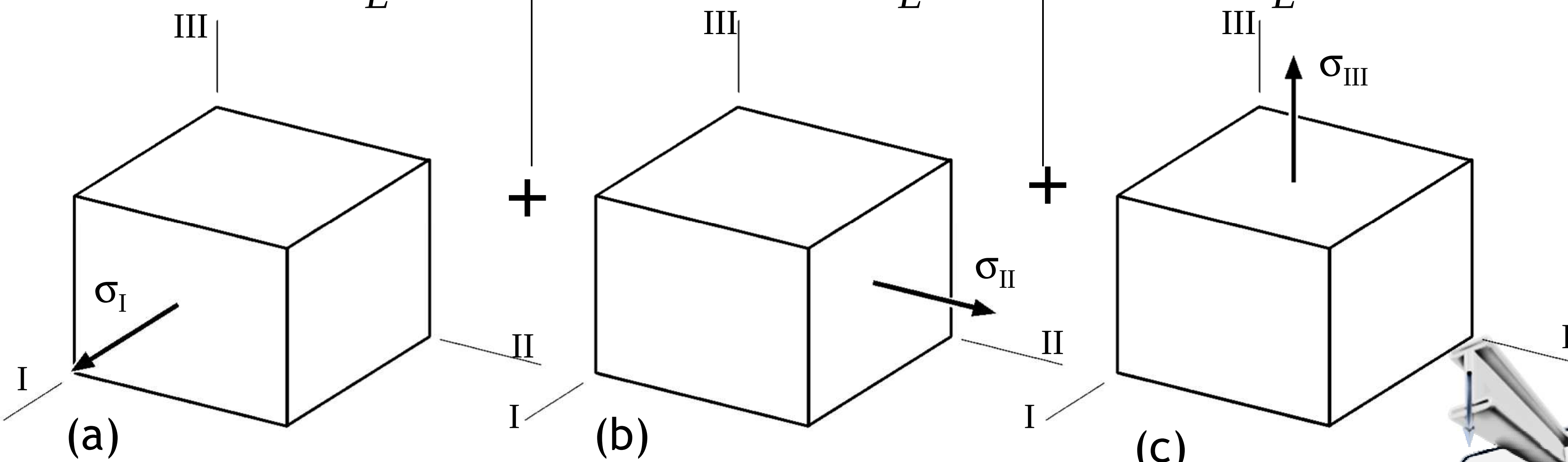
Ley de Hooke generalizada

$$\varepsilon_I = \frac{\sigma_I - \nu(\sigma_{II} + \sigma_{III})}{E}$$

$$\varepsilon_I = \frac{\sigma_I}{E} \qquad \varepsilon_{II} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E} \qquad \varepsilon_{III} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E} \qquad \varepsilon_{II} = \frac{\sigma_{II}}{E} \qquad \varepsilon_{III} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E} \qquad \varepsilon_{II} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E} \qquad \varepsilon_{III} = \frac{\sigma_{III}}{E}$$



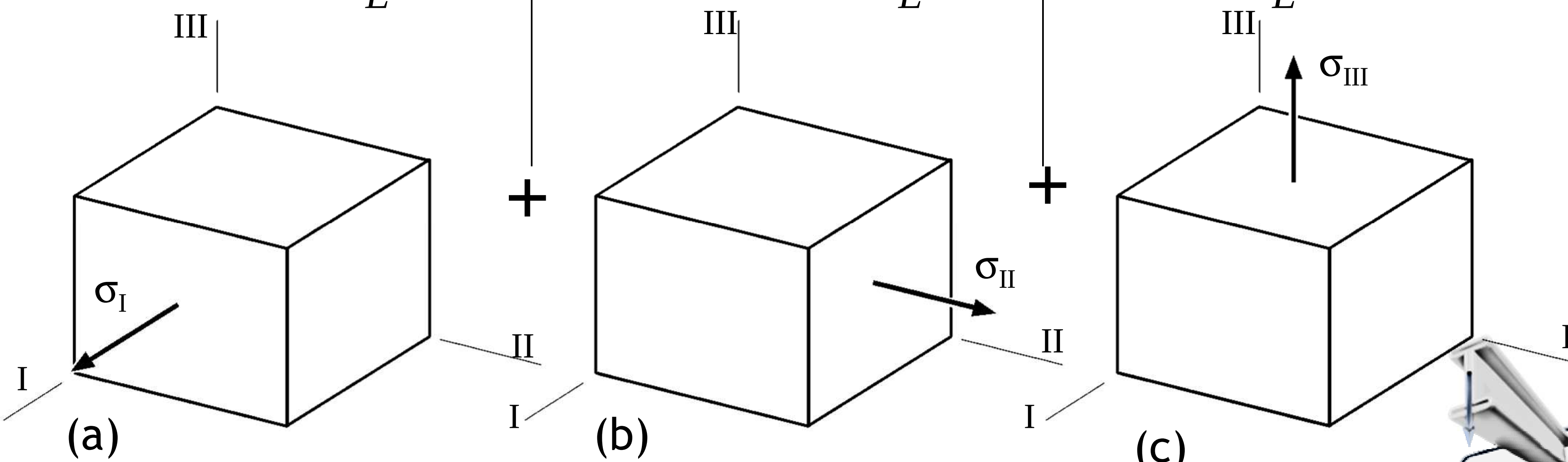
Ley de Hooke generalizada

$$\varepsilon_I = \frac{\sigma_I - \nu(\sigma_{II} + \sigma_{III})}{E} \qquad \varepsilon_{II} = \frac{\sigma_{II} - \nu(\sigma_I + \sigma_{III})}{E} \qquad \varepsilon_{III} = \frac{\sigma_{III} - \nu(\sigma_I + \sigma_{II})}{E}$$

$$\varepsilon_I = \frac{\sigma_I}{E} \qquad \varepsilon_{II} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E} \qquad \varepsilon_{III} = -\nu\varepsilon_I = -\nu\frac{\sigma_I}{E}$$

$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E} \qquad \varepsilon_{II} = \frac{\sigma_{II}}{E} \qquad \varepsilon_{III} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{II}}{E}$$

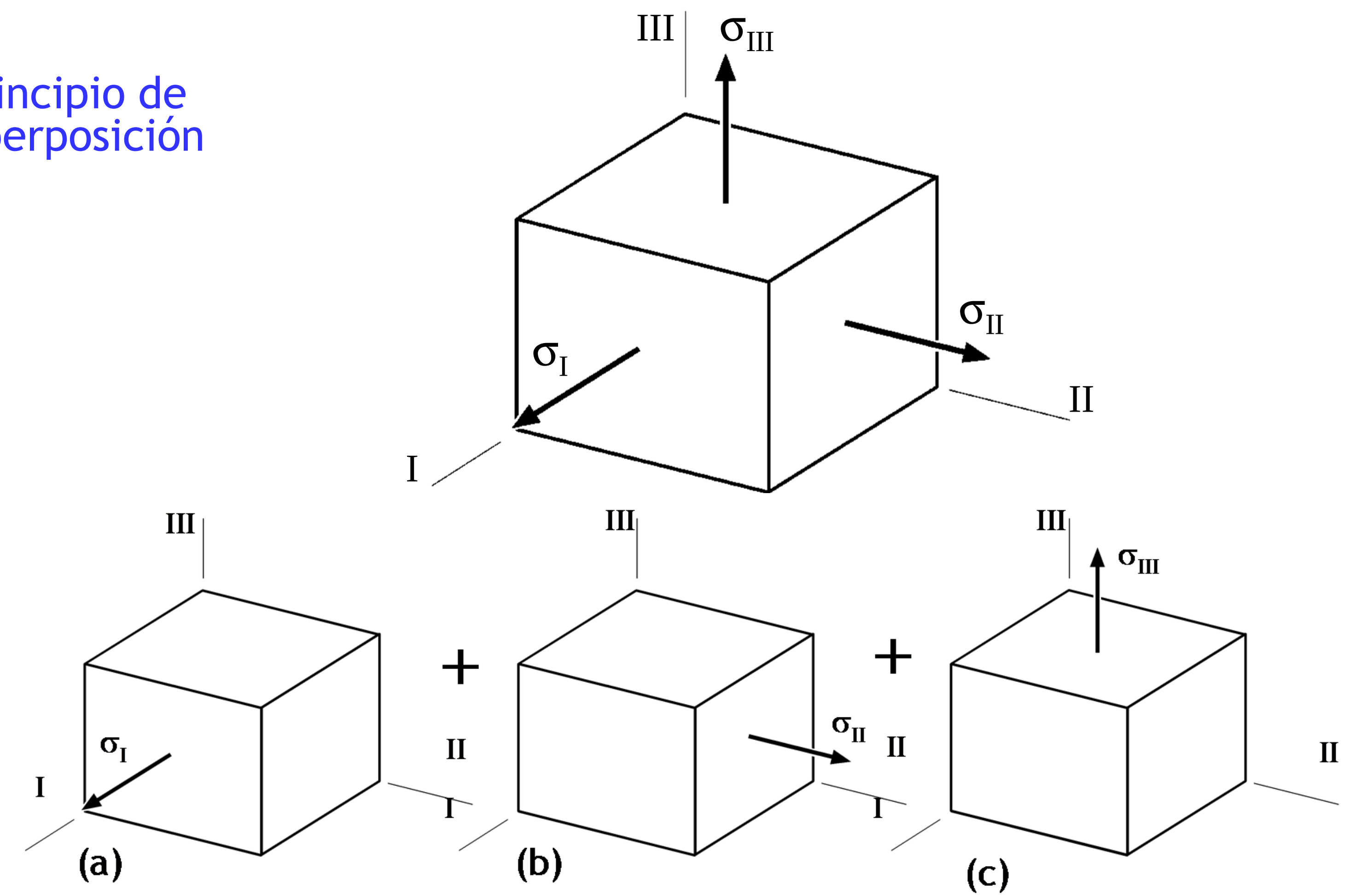
$$\varepsilon_I = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E} \qquad \varepsilon_{II} = -\nu\varepsilon_{II} = -\nu\frac{\sigma_{III}}{E} \qquad \varepsilon_{III} = \frac{\sigma_{III}}{E}$$



Ley de Hooke generalizada

$$\varepsilon_I = \frac{\sigma_I - \nu\left(\sigma_{II} + \sigma_{III}\right)}{E} \qquad \varepsilon_{II} = \frac{\sigma_{II} - \nu\left(\sigma_I + \sigma_{III}\right)}{E} \qquad \varepsilon_{III} = \frac{\sigma_{III} - \nu\left(\sigma_I + \sigma_{II}\right)}{E}$$

Principio de superposición



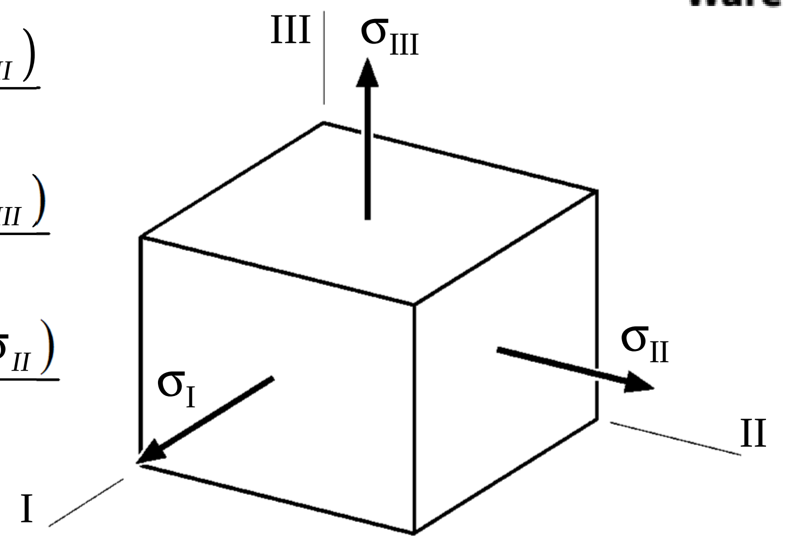
Módulo de cizalladura

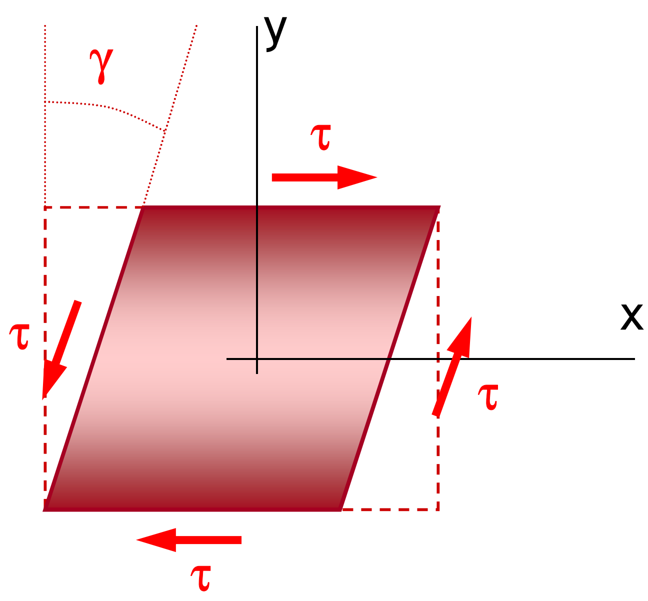
Relaciones tensión/deformación normales

$$\varepsilon_I = \frac{\sigma_I - \nu(\sigma_{II} + \sigma_{III})}{E}$$

$$\varepsilon_{II} = \frac{\sigma_{II} - \nu(\sigma_I + \sigma_{III})}{E}$$

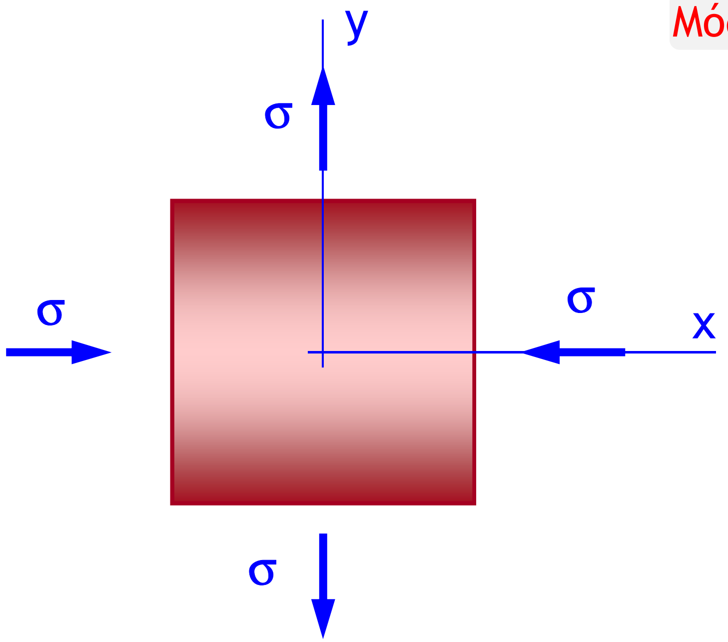
$$\varepsilon_{III} = \frac{\sigma_{III} - \nu(\sigma_I + \sigma_{II})}{E}$$

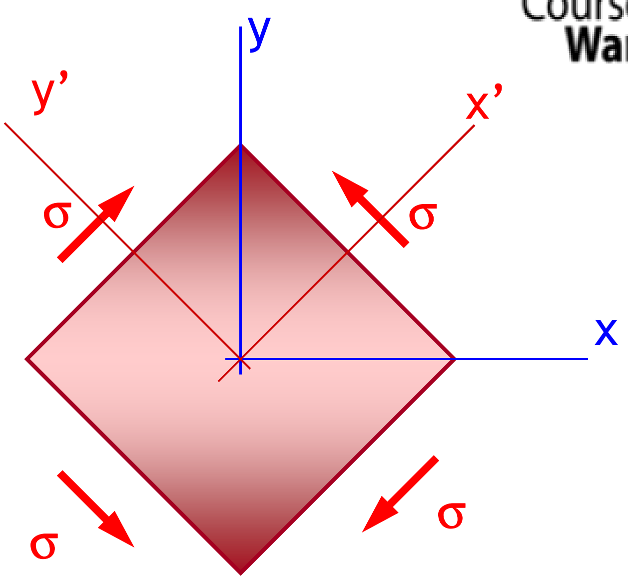




¿Qué ocurre con las componentes tangenciales y las deformaciones angulares?

Módulo de cizalladura



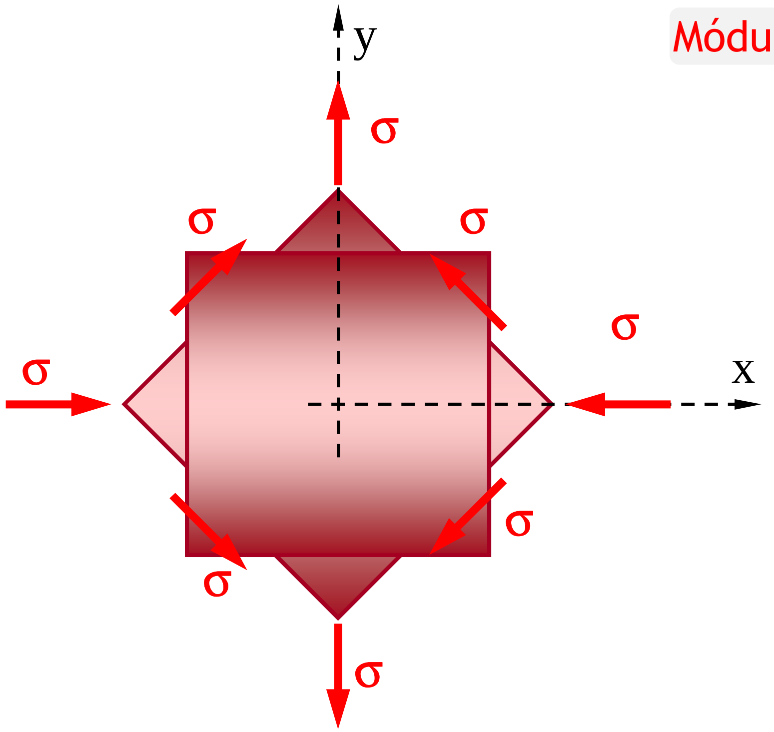


MISMO
estado tensional
utilizando un
sistema de referencia
diferente

$$\boldsymbol{\sigma} = \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

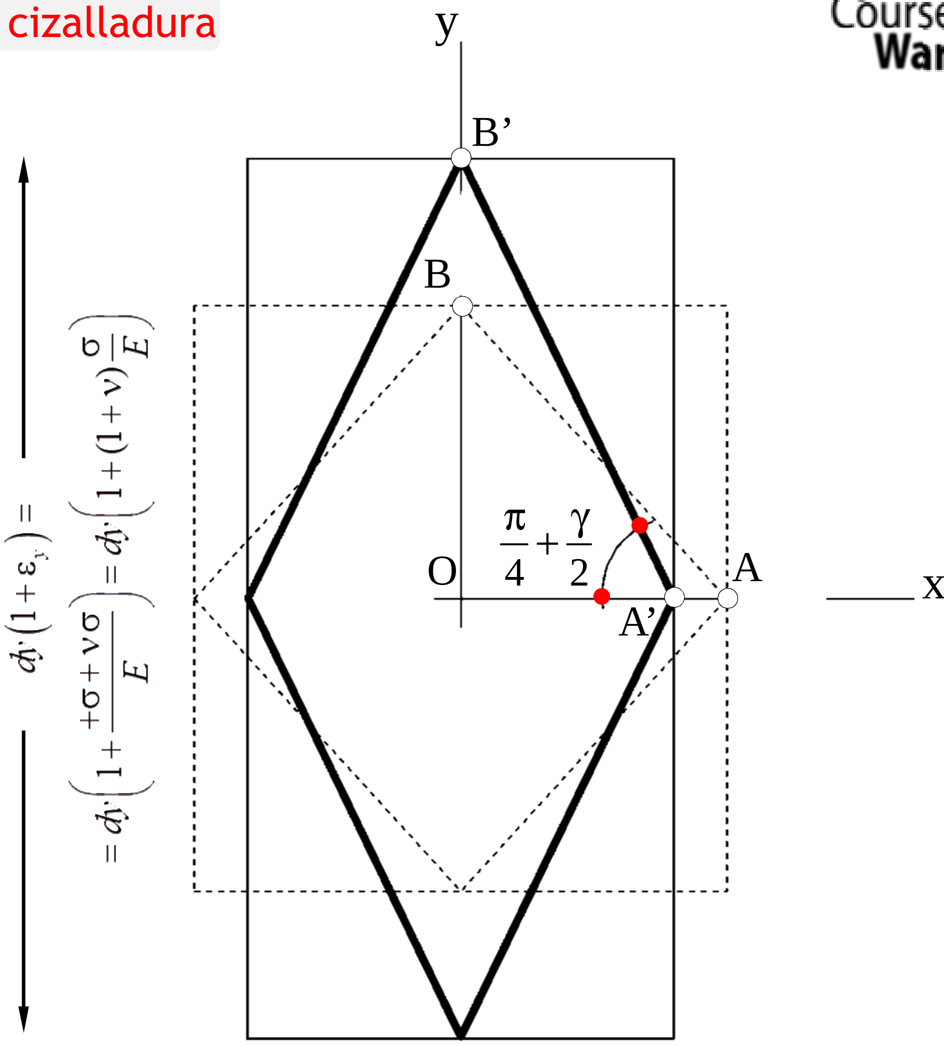
$$\boldsymbol{\sigma}' = \begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Módulo de cizalladura



$$\varepsilon_x = \frac{-\sigma - \nu\sigma}{E} = -(1+\nu)\frac{\sigma}{E}$$

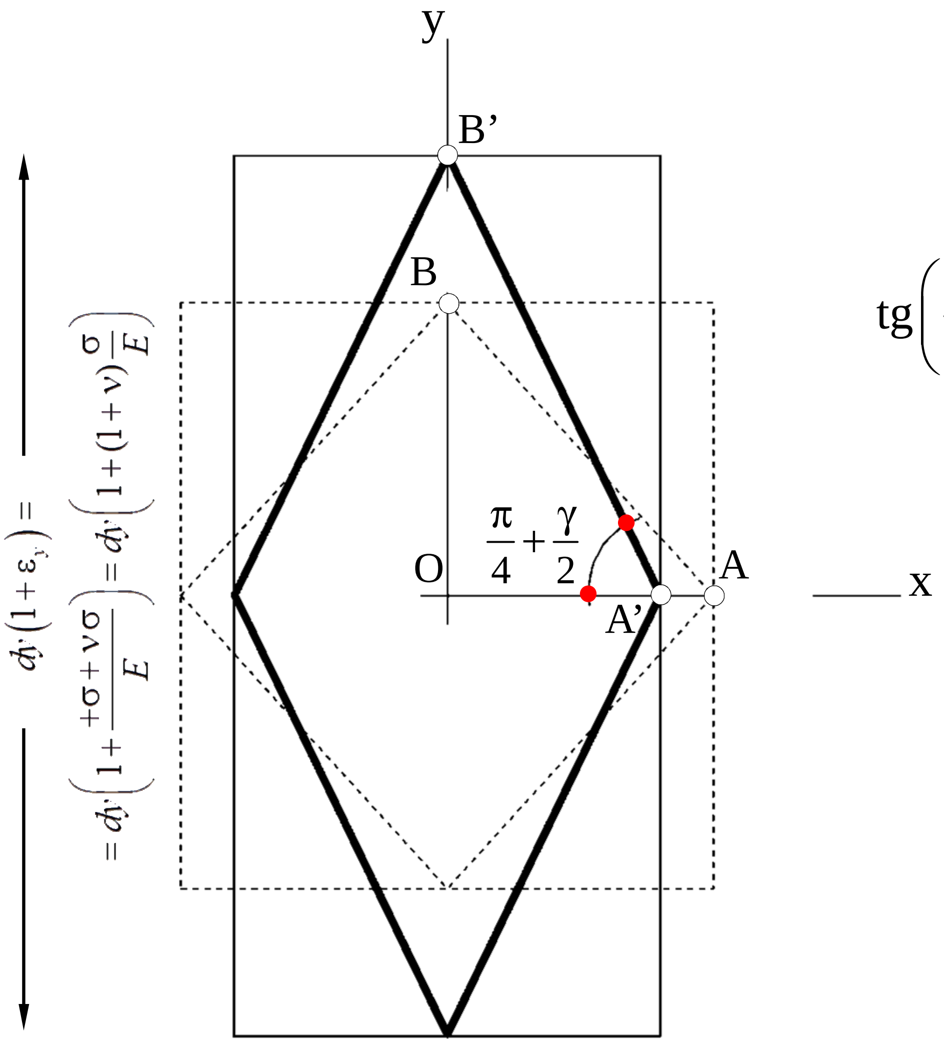
$$\varepsilon_y = \frac{\sigma - \nu(-\sigma)}{E} = (1+\nu)\frac{\sigma}{E}$$



$$dy\,(1+\varepsilon_y) = \\ = dy\left(1+\frac{+\sigma+\nu\sigma}{E}\right) = dy\left(1+(1+\nu)\frac{\sigma}{E}\right)$$

$$dx\,(1+\varepsilon_x) = \\ = dx\left(1+\frac{-\sigma-\nu\sigma}{E}\right) = dx\left(1-(1+\nu)\frac{\sigma}{E}\right)$$

Módulo de cizalladura



$$dx\left(1+\varepsilon_x\right)=$$

$$=dx\left(1+\frac{-\sigma-\nu\sigma}{E}\right)=dx\left(1-(1+\nu)\frac{\sigma}{E}\right)$$

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)=\frac{1+\varepsilon_y}{1+\varepsilon_x}=\frac{1+(1+\nu)\sigma/E}{1-(1+\nu)\sigma/E}$$

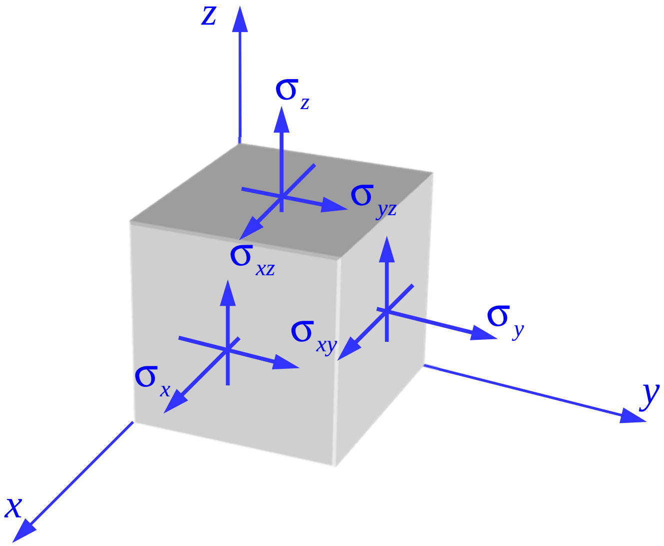
$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)=\frac{\operatorname{tg}(\pi/4)+\operatorname{tg}(\gamma/2)}{1-\operatorname{tg}(\pi/4)\operatorname{tg}(\gamma/2)}\approx\frac{1+\gamma/2}{1-\gamma/2}$$

$$\frac{\gamma}{2}=\frac{(1+\nu)}{E}\sigma:=\frac{\sigma}{2G}$$

$$G=\frac{E}{2(1+\nu)}$$

Nota: $G>0$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \quad (2)$$

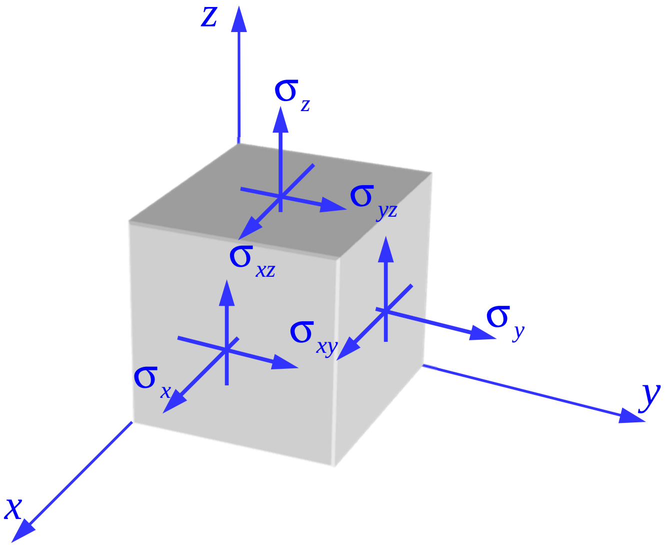
$$\varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \quad (3)$$

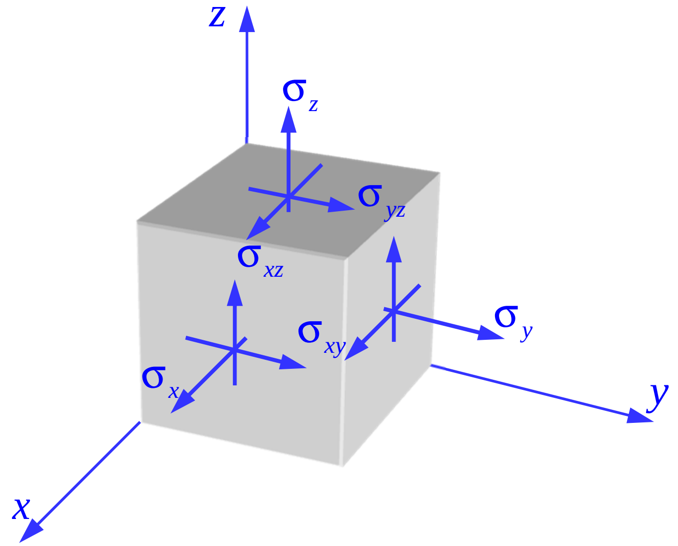
$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

Dirección principal deformación = Dirección principal tensión

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

---

(1)+(2)+(3)

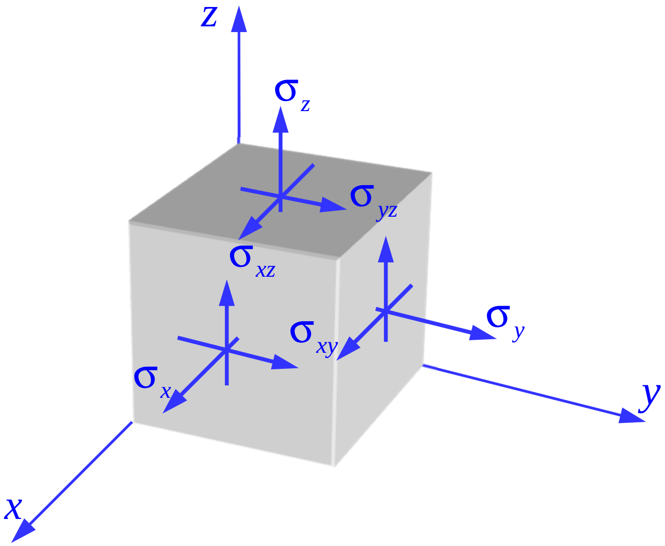
$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z - 2\nu\left(\sigma_x + \sigma_y + \sigma_z\right)}{E}$$

$$e = \frac{1-2\nu}{E} I_1$$

El tensor octaédrico de tensión es el responsable del cambio de volumen

$$\boldsymbol{\sigma} = \frac{I_1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

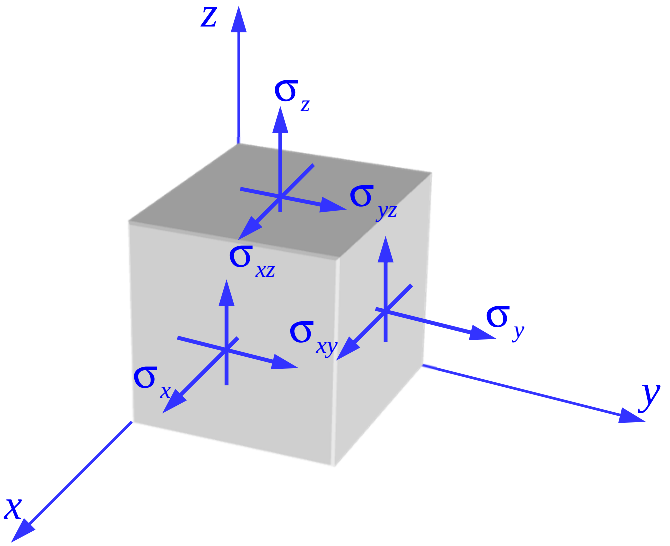
(1)+(2)+(3)

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z - 2\nu(\sigma_x + \sigma_y + \sigma_z)}{E} \Rightarrow e = \frac{1-2\nu}{E} I_1$$

$$\boldsymbol{\sigma} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\frac{\Delta V}{V} = -3\frac{1-2\nu}{E}P \Rightarrow P = -\frac{E}{3(1-2\nu)}\frac{\Delta V}{V}$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \qquad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \qquad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \qquad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \qquad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \qquad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \qquad (7)$$

(1)+(2)+(3)

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z - 2\nu\left(\sigma_x + \sigma_y + \sigma_z\right)}{E}$$

$$e = \frac{1-2\nu}{E} I_1$$

$$\boldsymbol{\sigma} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\frac{\Delta V}{V} = -3\frac{1-2\nu}{E}P$$

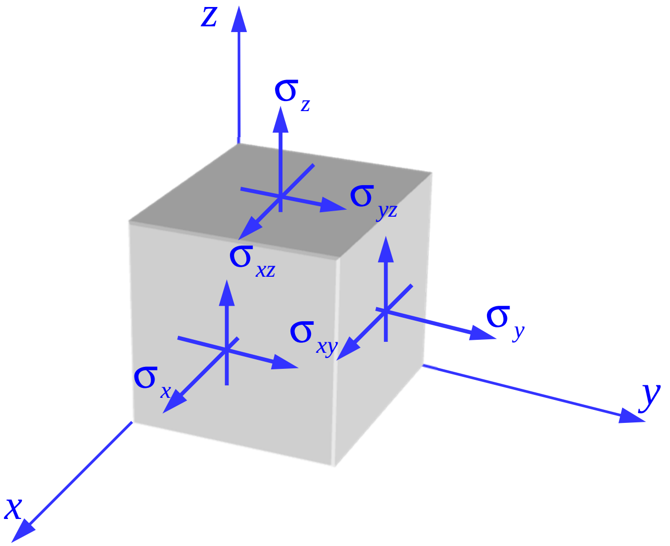
$$P = -\frac{E}{3(1-2\nu)}\frac{\Delta V}{V}$$

$$\mathbf{F} = K\mathbf{u}$$

Módulo de rigidez volumétrica

$$K = \frac{E}{3(1-2\nu)}$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

---

(1)+(2)+(3)

$$e = \frac{1-2\nu}{E} I_1$$

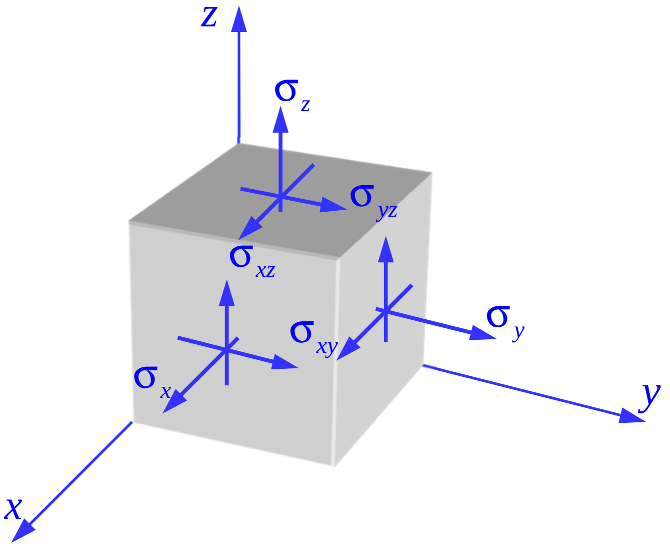
$$\boldsymbol{\sigma} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} \Rightarrow P = -\frac{E}{3(1-2\nu)}\frac{\Delta V}{V} \Rightarrow K = \frac{E}{3(1-2\nu)}$$

Módulo de rigidez volumétrica

Nota: $K > 0 \iff 1-2\nu > 0 \iff 1/2 > \nu \iff 0 < \nu < 1/2$

$$\nu = 1/2 \Rightarrow K = \infty \Rightarrow \frac{\Delta V}{V} = -\frac{P}{K} = 0 \Rightarrow V\ cte$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \quad (3)$$

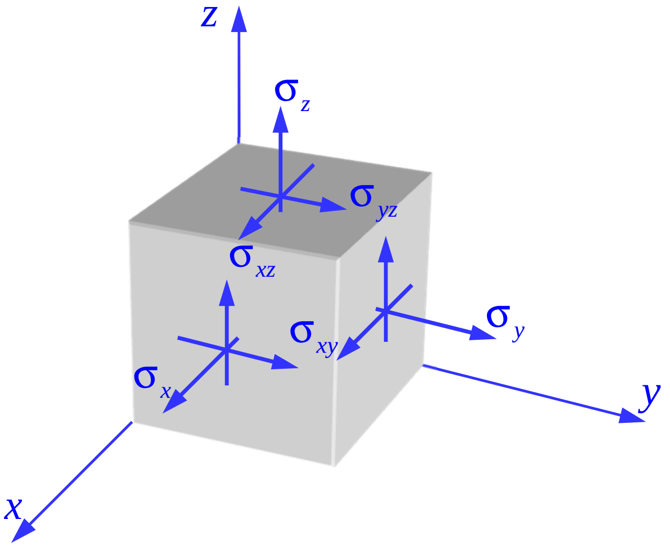
$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

$$(1) \quad \varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_x + \sigma_y + \sigma_z\right) + \nu\sigma_x}{E} \Rightarrow \varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E} I_1$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

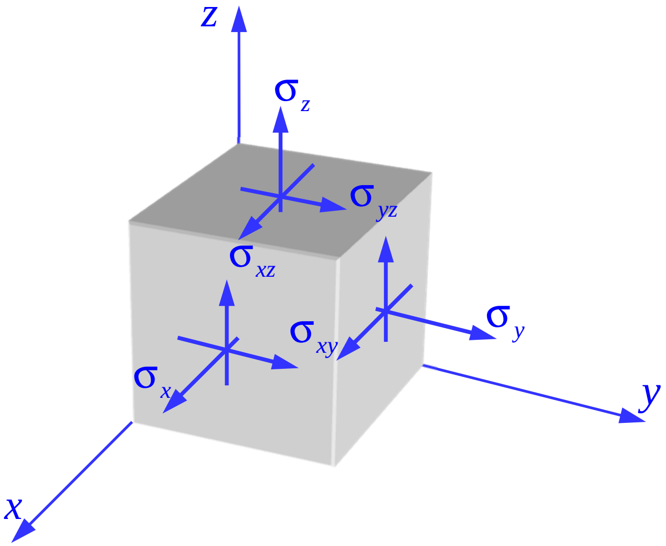
$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

$$(1) \quad \varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_x + \sigma_y + \sigma_z\right) + \nu\sigma_x}{E} \Rightarrow \varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}I_1 \Rightarrow$$

$$\Rightarrow \sigma_x = \left(\frac{E}{1+\nu}\varepsilon_x + \frac{\nu E}{(1-2\nu)(1+\nu)}e\right)$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

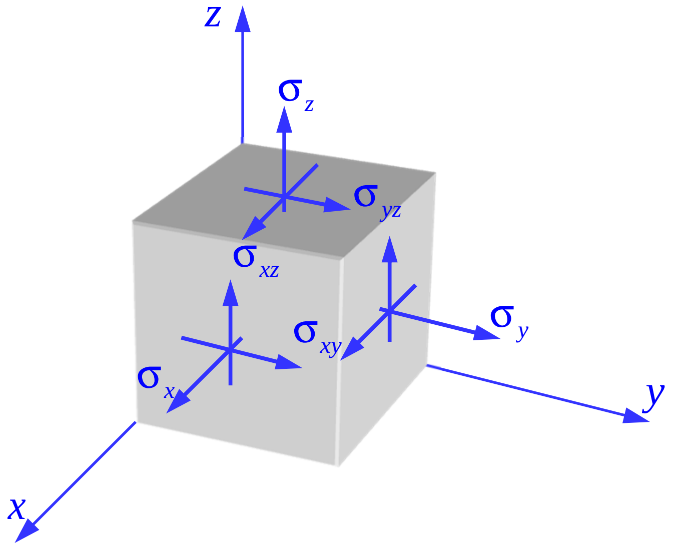
$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

$$(1) \quad \varepsilon_x = \frac{\sigma_x - \nu(\sigma_x + \sigma_y + \sigma_z) + \nu\sigma_x}{E} \Rightarrow \varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}I_1 \Rightarrow$$

$$\Rightarrow \sigma_x = \left(\underbrace{\frac{E}{1+\nu}}_{2G}\varepsilon_x + \frac{\nu E}{(1-2\nu)(1+\nu)}e\right)$$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

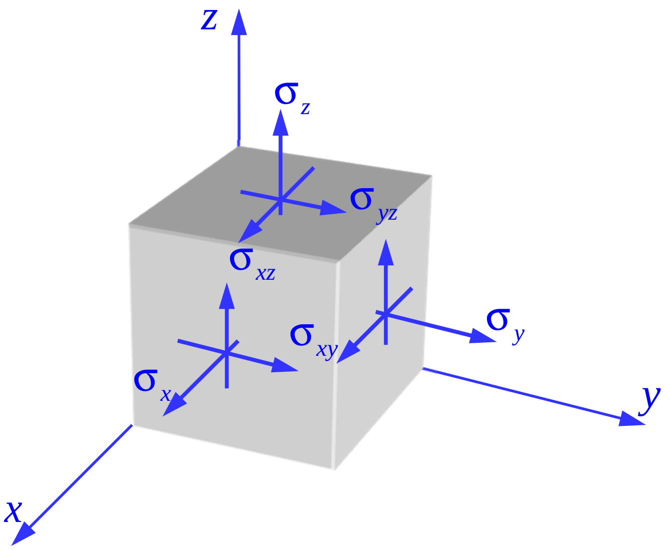
$$(1) \quad \varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_x + \sigma_y + \sigma_z\right) + \nu\sigma_x}{E} \Rightarrow \varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E} I_1 \Rightarrow$$

$$\Rightarrow \sigma_x = \left(\underbrace{\frac{E}{1+\nu}}_{2G} \varepsilon_x + \underbrace{\frac{\nu E}{(1-2\nu)(1+\nu)}}_{\lambda} e \right)$$

Constante de Lamé

Nota: $\lambda > 0$

Ley de comportamiento en coordenadas cualesquiera



$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \quad (2)$$

$$\varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E} \quad (3)$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \quad (5)$$

$$\varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz} \quad (7)$$

Ecs. de Hooke

$$\sigma_x = 2G\varepsilon_x + \lambda e \quad (8)$$

$$\sigma_y = 2G\varepsilon_y + \lambda e \quad (9)$$

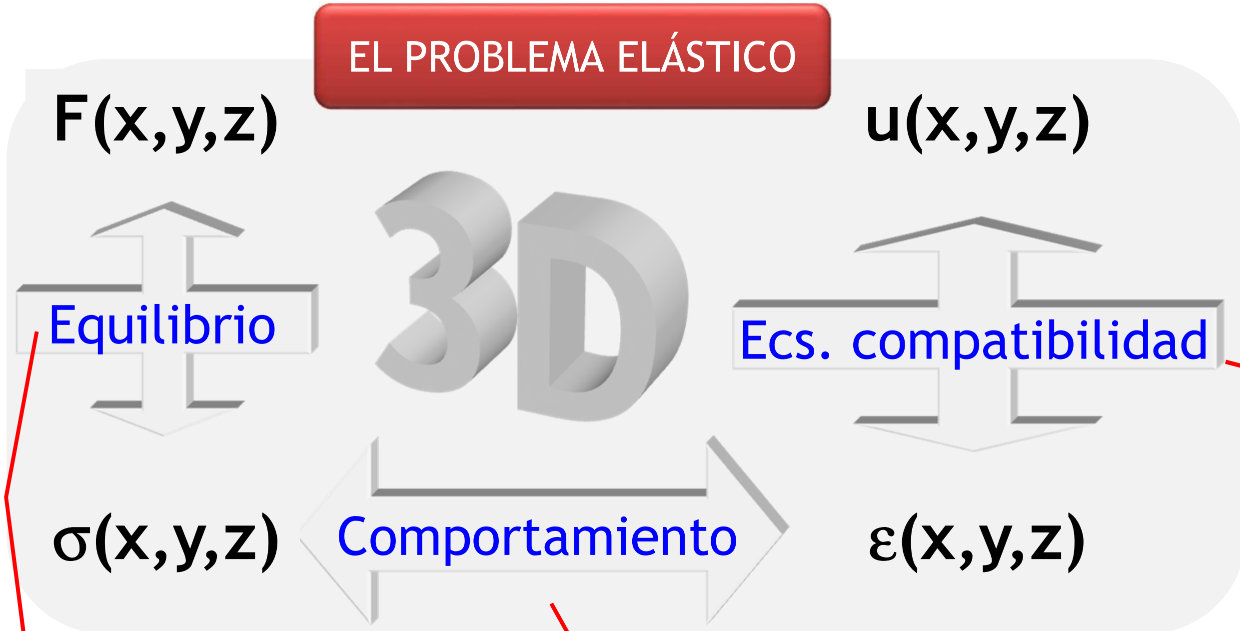
$$\sigma_z = 2G\varepsilon_z + \lambda e \quad (10)$$

Ecs. de Lamé

$$\sigma_{xy} = 2G\varepsilon_{xy} \quad (11)$$

$$\sigma_{xz} = 2G\varepsilon_{xz} \quad (12)$$

$$\sigma_{yz} = 2G\varepsilon_{yz} \quad (13)$$



$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}\right)$$

$$\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2}\right)$$

$$\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xz}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yx}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{zy}}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

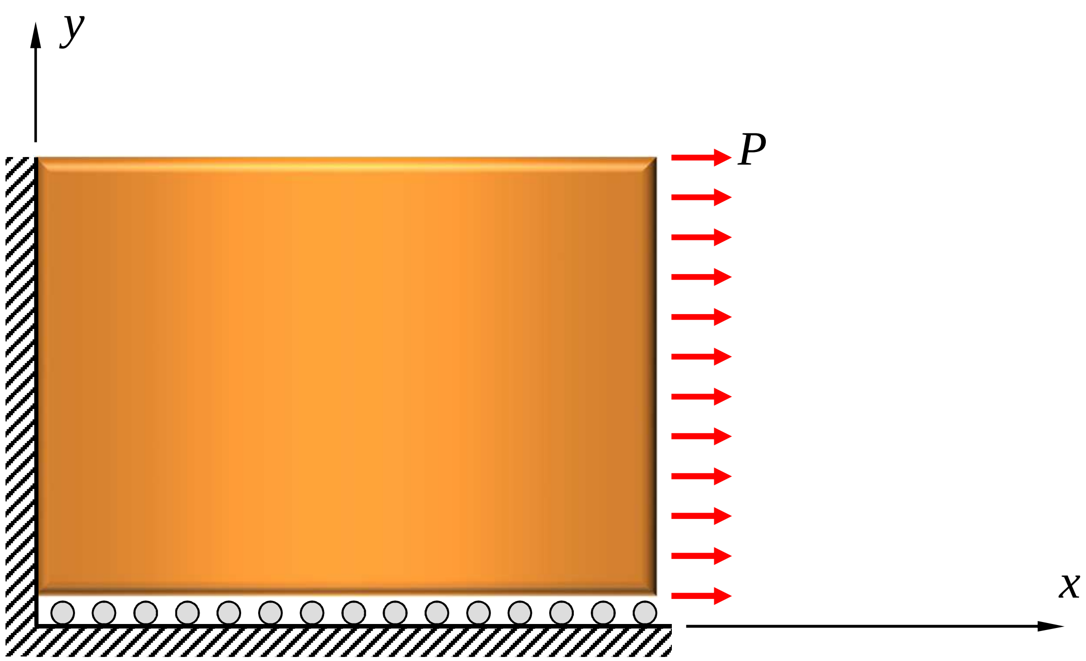
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0$$

$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \qquad \varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy}$$

$$\varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \qquad \varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz}$$

$$\varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz}$$

Condiciones de contorno

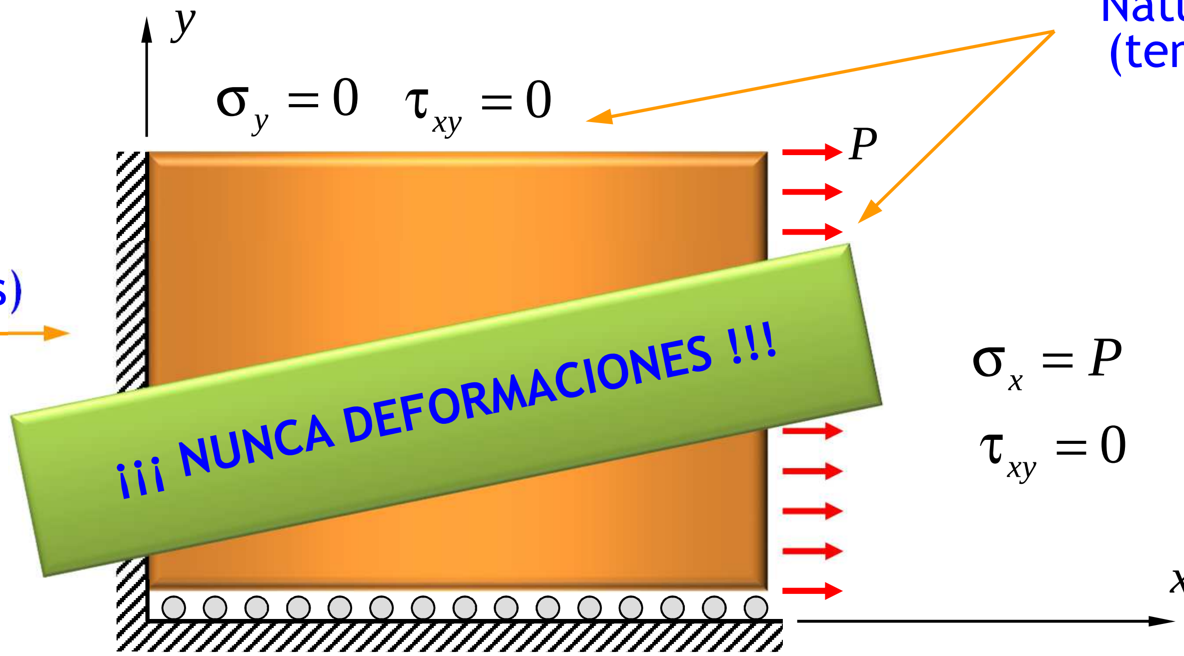


Condiciones de contorno

Naturales
(tensión)

Esenciales
(desplazamientos)

$$u_x = 0$$
$$u_y = 0$$

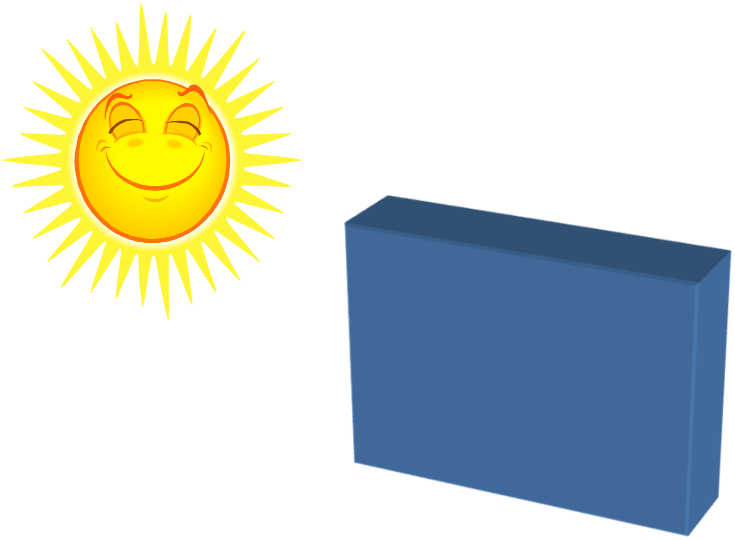


$$\sigma_x = P$$
$$\tau_{xy} = 0$$

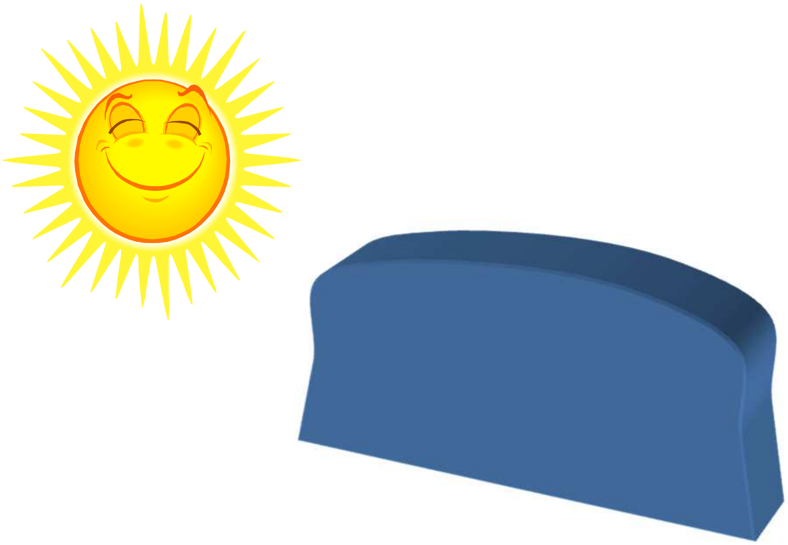
Mixtas

$$\tau_{xy} = 0$$
$$u_y = 0$$

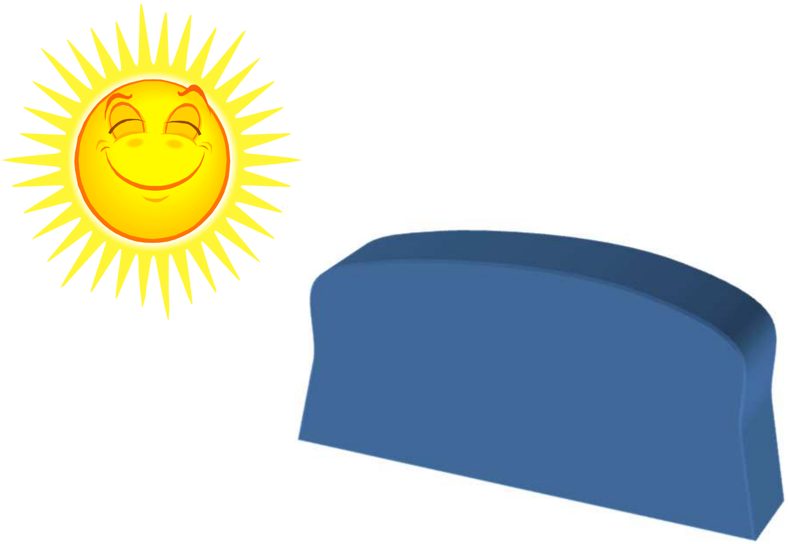
El problema térmico. Una primera aproximación



El problema térmico. Una primera aproximación



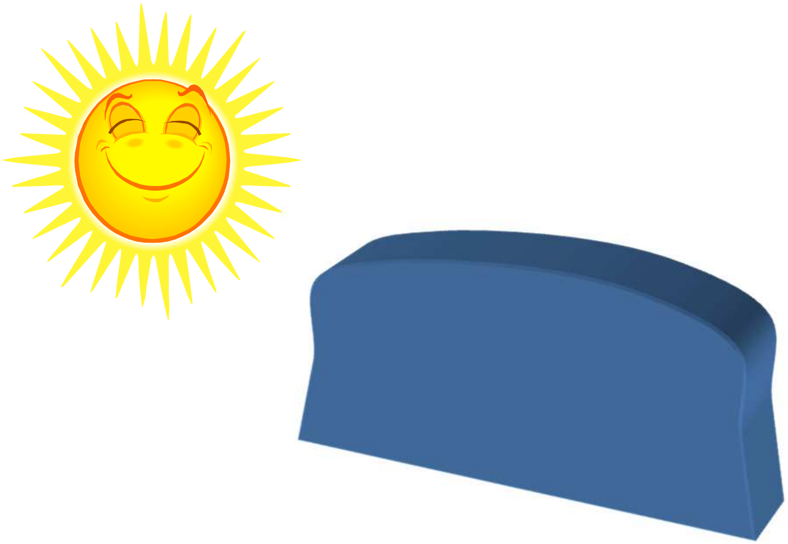
El problema térmico. Una primera aproximación



Si $\alpha \neq f(T)$

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{\alpha l_0 \Delta T}{l_0} \quad \Rightarrow \quad \begin{cases} \varepsilon_x = \alpha \Delta T \\ \varepsilon_y = \alpha \Delta T \\ \varepsilon_z = \alpha \Delta T \end{cases}$$

El problema térmico. Una primera aproximación



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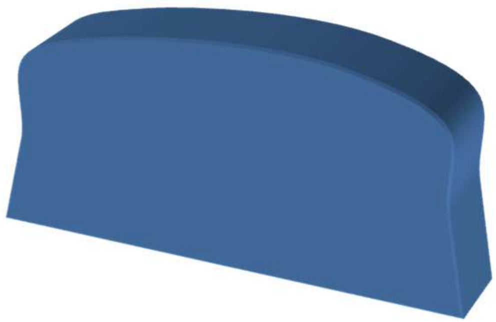
$$\varepsilon = \frac{\Delta l}{l_0} = \frac{\alpha l_0 \Delta T}{l_0} \Rightarrow \begin{cases} \varepsilon_x = \alpha \Delta T \\ \varepsilon_y = \alpha \Delta T \\ \varepsilon_z = \alpha \Delta T \end{cases}$$

Si no hay restricciones al movimiento

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \alpha\Delta T & 0 & 0 \\ 0 & \alpha\Delta T & 0 \\ 0 & 0 & \alpha\Delta T \end{pmatrix}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

El problema térmico. Una primera aproximación



Si $\alpha \neq f(T)$

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{\alpha l_0 \Delta T}{l_0} \Rightarrow \begin{cases} \varepsilon_x = \alpha \Delta T \\ \varepsilon_y = \alpha \Delta T \\ \varepsilon_z = \alpha \Delta T \end{cases}$$

Si no hay restricciones al movimiento

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \alpha\Delta T & 0 & 0 \\ 0 & \alpha\Delta T & 0 \\ 0 & 0 & \alpha\Delta T \end{pmatrix}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T = T(x, y, z)$$

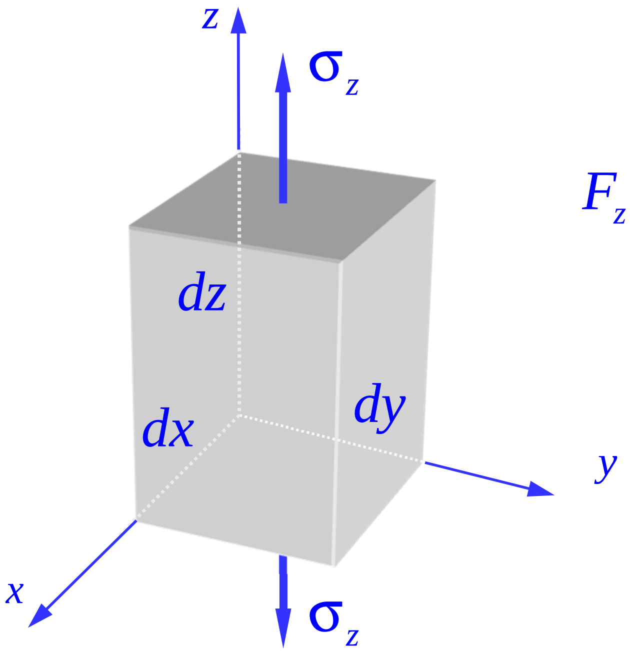
$$\Downarrow$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{mec} + \boldsymbol{\varepsilon}^{ter}$$

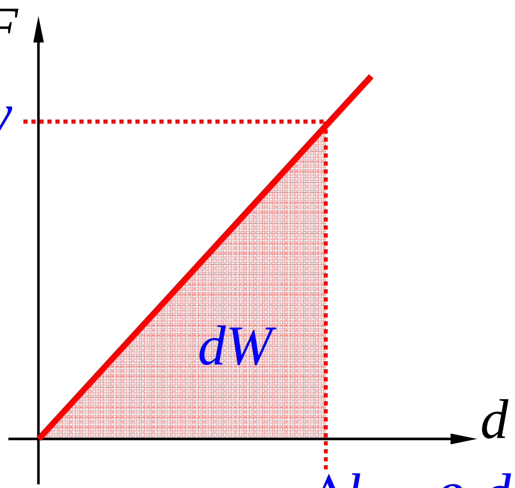
Energía de deformación

$$\varepsilon_z = \sigma_z / E \quad \Leftrightarrow \quad \Delta l_z = \varepsilon_z dz$$

$$\sigma_z \quad \Leftrightarrow \quad F_z = \sigma_z dxdy$$



$$F_z = \sigma_z dxdy$$



$$\Delta l_z = \varepsilon_z dz = \frac{\sigma_z}{E} dz$$

$$dW = \frac{1}{2} F \Delta l_z = \frac{1}{2} \sigma_z \varepsilon_z dxdydz = \frac{1}{2} \sigma_z \varepsilon_z dV$$

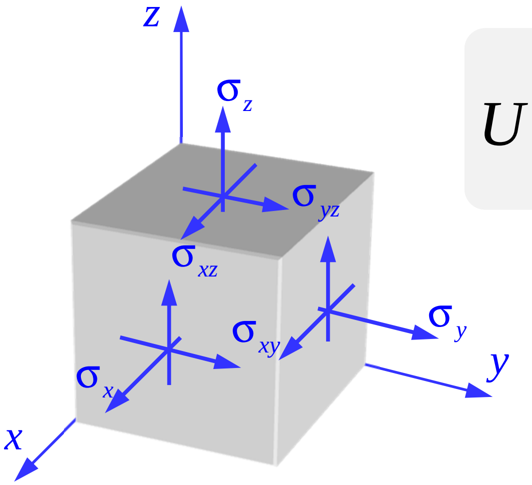
Energía unitaria de deformación

Energía de deformación por unidad de volumen

Energía de deformación

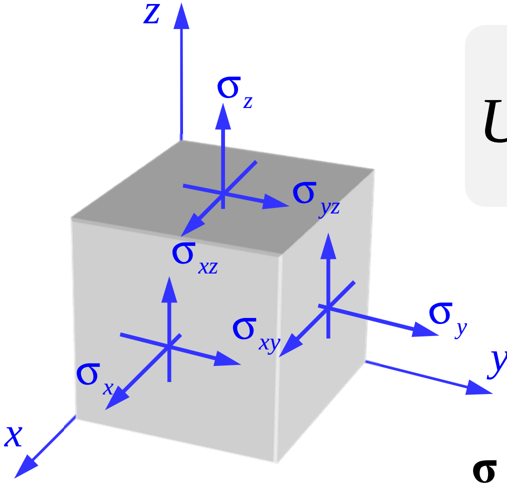
$$\frac{dW}{dV} = \frac{1}{2} \sigma_z \varepsilon_z := U$$

Energía de deformación



$$U = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)$$

Energía de deformación

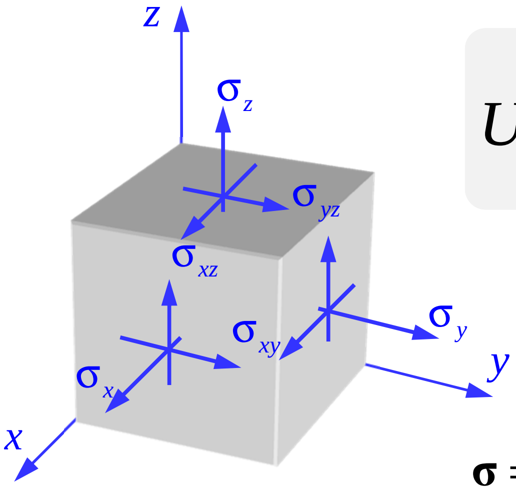


$$U = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)$$

$$\left.\begin{array}{l} \boldsymbol{\sigma} = \dfrac{I_1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \boldsymbol{\varepsilon} = \dfrac{e}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}\right\} \Rightarrow \quad U_0 = \frac{eI_1}{6}$$

Energía de cambio de volumen

Energía de deformación



$$U=\frac{1}{2}\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{xz}\gamma_{xz}+\tau_{yz}\gamma_{yz}\right)$$

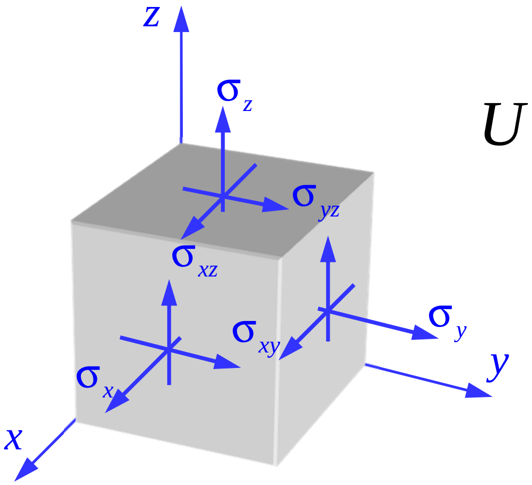
$$\left.\begin{array}{l}\boldsymbol{\sigma}=\dfrac{I_1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\\[2ex]\boldsymbol{\varepsilon}=\dfrac{e}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\end{array}\right\}\Rightarrow U_0=\frac{eI_1}{6}$$

Energía de cambio de volumen

Energía de cambio de forma

$$U_d=U-U_0=\frac{1+\nu}{6E}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_x-\sigma_z\right)^2+\left(\sigma_y-\sigma_z\right)^2\right]+\frac{1}{2G}\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)$$

Energía de deformación

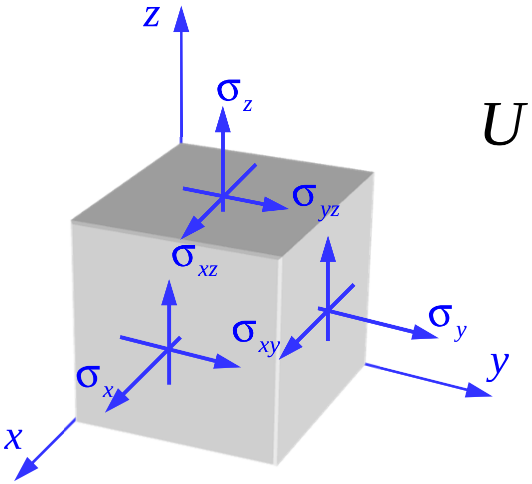


$$U = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)$$

En ausencia de fenómenos disipativos

Energía potencial

Energía de deformación



$$U = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)$$

En ausencia de fenómenos disipativos

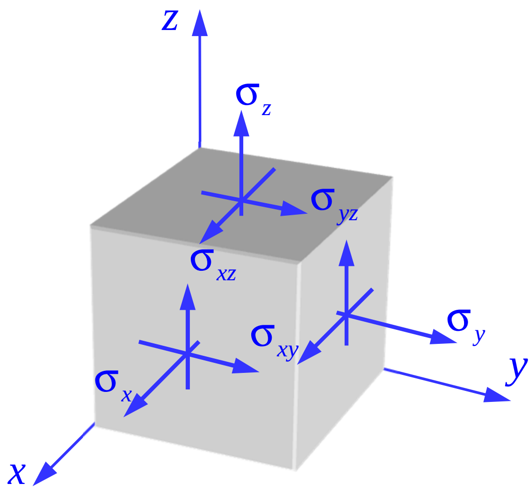
Energía potencial

Energía potencial total

$$E_p = \frac{1}{2}\int\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)dV$$

Igual a la energía total de deformación e igual al trabajo externo

Tensión plana vs. Deformación plana



Ecs. de Hooke

$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \qquad \varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \qquad \varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E}$$

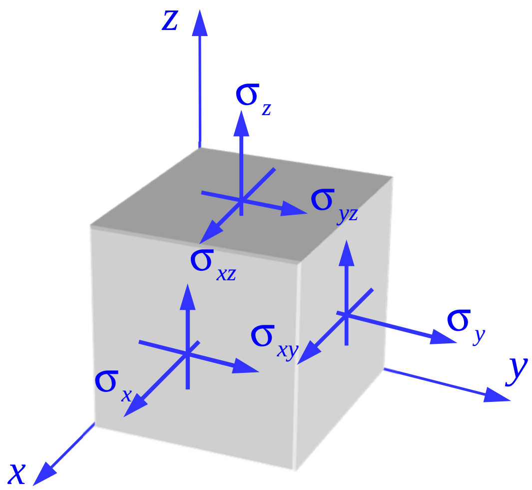
$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \qquad \varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz}$$

Tensión Plana

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & -\nu(\sigma_x + \sigma_y)/E \end{pmatrix}$$

Deformación Plana $\Leftrightarrow$ $\sigma_x = -\sigma_y$

Tensión plana vs. Deformación plana



Ecs. de Hooke

$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \qquad \varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \qquad \varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E}$$

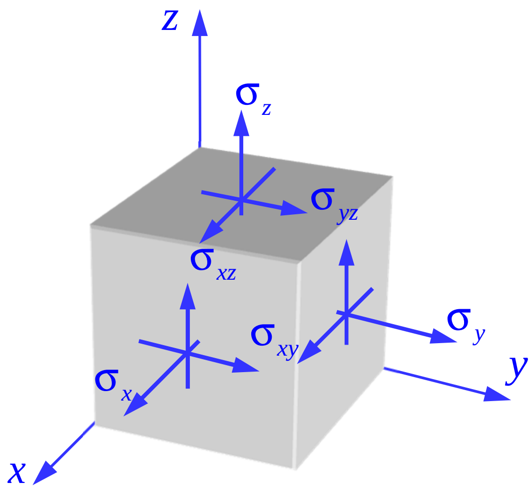
$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \qquad \varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz}$$

Tensión Plana

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & -\nu(\sigma_x + \sigma_y)/E \end{pmatrix}$$

Deformación Plana $\Leftrightarrow \quad \sigma_x = -\sigma_y$

Tensión plana vs. Deformación plana



Ecs. de Hooke

$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E} \qquad \varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_x + \sigma_z\right)}{E} \qquad \varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E}$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \qquad \varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz}$$

Tensión Plana

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & -\nu\left(\sigma_x + \sigma_y\right)/E \end{pmatrix}$$

Deformación Plana $\Leftrightarrow \sigma_x = -\sigma_y$

Deformación Plana

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & \lambda(\varepsilon_x + \varepsilon_x) \end{pmatrix}$$

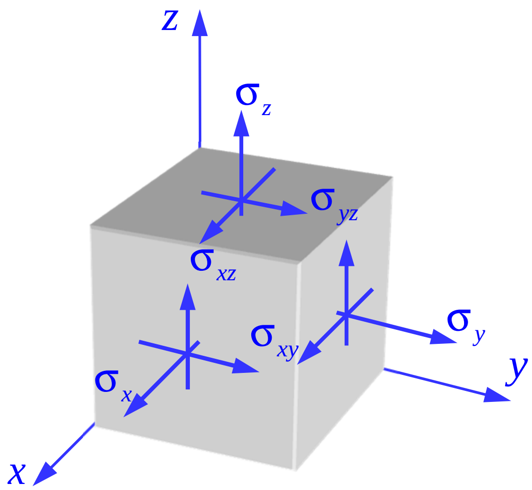
Tensión Plana $\Leftrightarrow \varepsilon_x = -\varepsilon_y$

Ecs. de Lamé

$$\sigma_x = 2G\varepsilon_x + \lambda e \qquad \sigma_y = 2G\varepsilon_y + \lambda e \qquad \sigma_z = 2G\varepsilon_z + \lambda e$$

$$\sigma_{xy} = 2G\varepsilon_{xy} \qquad \sigma_{xz} = 2G\varepsilon_{xz} \qquad \sigma_{yz} = 2G\varepsilon_{yz}$$

Tensión plana vs. Deformación plana



Ecs. de Hooke

$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} \qquad \varepsilon_y = \frac{\sigma_y - \nu(\sigma_x + \sigma_z)}{E} \qquad \varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E}$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy} \qquad \varepsilon_{xz} = \frac{(1+\nu)}{E}\sigma_{xz} \qquad \varepsilon_{yz} = \frac{(1+\nu)}{E}\sigma_{yz}$$

Tensión Plana

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & -\nu(\sigma_x + \sigma_y)/E \end{pmatrix}$$

Deformación Plana $\Leftrightarrow \sigma_x = -\sigma_y$

Deformación Plana

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y & 0 \\ 0 & 0 & \lambda(\varepsilon_x + \varepsilon_x) \end{pmatrix}$$

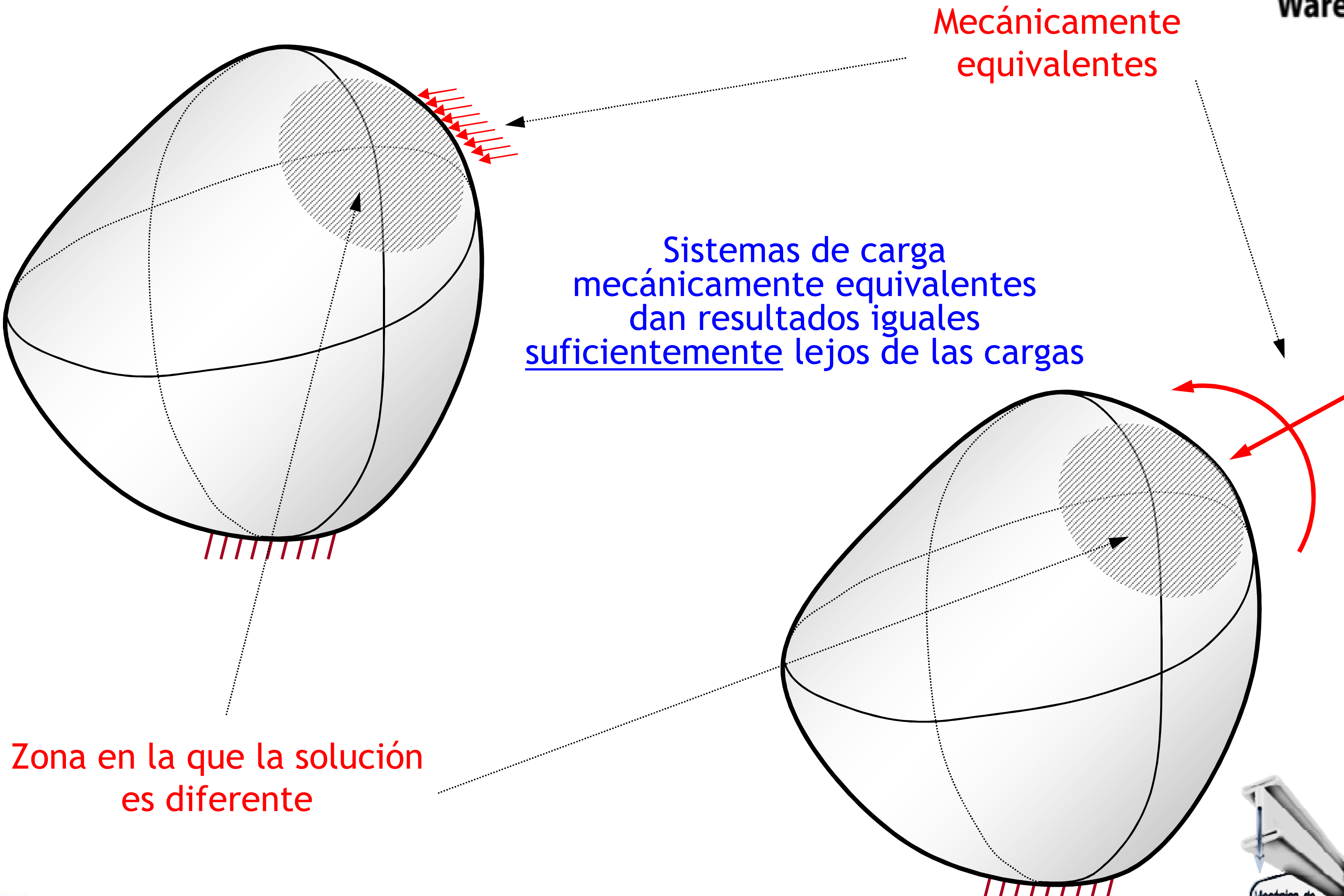
Tensión Plana $\Leftrightarrow \varepsilon_x = -\varepsilon_y$

Ecs. de Lamé

$$\sigma_x = 2G\varepsilon_x + \lambda e \qquad \sigma_y = 2G\varepsilon_y + \lambda e \qquad \sigma_z = 2G\varepsilon_z + \lambda e$$

$$\sigma_{xy} = 2G\varepsilon_{xy} \qquad \sigma_{xz} = 2G\varepsilon_{xz} \qquad \sigma_{yz} = 2G\varepsilon_{yz}$$

Principio de Saint-Venant



Principio de Saint-Venant